CARL VON OSSIETZKY universität OLDENBURG

Oldenburg Discussion Papers in Economics

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V – 405-17

November 2017

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From constrained optimization to constrained dynamics: extending analogies between economics and mechanics

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Keywords: Economic Model; Economic dynamics; Disequilibrium economics; Classical Mechanics; Constrained Optimization; Constrained Dynamics; General Equilibrium; Edgeworth Box. **JEL:** A12; B00; C62; D50.

Abstract: Since the beginnings of modern economics, economists sought to emulate the revolution in physics initiated by Newton's 'Principia' (1687). Concepts from mechanics have influenced economic models both in terms of methodology and content. The aim of this theoretical paper is to put forward a novel economic modeling framework that extends the analogies between economics and classical mechanics from constrained optimization to constrained dynamics. We introduce the concepts of economic forces and economic power that bear striking resemblance to physical forces and the reciprocal value of mass. In this setup, the change of a variable is determined by the forces agents employ to change it according to their desire, their power to assert their interest, and constraint forces emerging from system constraints. The approach is based on a genuine dynamic out-of-equilibrium analysis and can incorporate heterogeneous agents, prisoner's dilemma situations, and behavioral assumptions different from rationality and utility maximization. Thereby, it seeks to overcome some restrictions inherent to approaches based on optimization under constraint and provide an out-of-equilibrium foundation for equilibrium models. We transform a static textbook exchange model into a dynamic model, and reflect on advantages, extensions and caveats of our modeling approach.

1 Introduction

Inspired by the description of stationary states in mechanical models (section 2), modern economics has been shaped by models drawing on the principles of optimization under constraint and the general equilibrium. This paper argues that the versatile dynamic theory of interaction provided by classical mechanics can be advantageous even beyond equilibrium approaches.

We put forward a new genuinely dynamic modeling framework that expands existing analogies to mechanical systems from *constrained optimization* to *general constrained dynamics* in continuous time (section 3). Based on an analogy of interacting 'bodies' under constraint from mechanics, the modeling approach depicts the economy from the perspective of *economic forces* and *economic power*, similar to the concepts of physical forces and the reciprocal value of mass. Economic force corresponds to the desire of agents to change certain variables, while economic power captures their ability to assert their interest to change them. The introduction of constraint forces, i. e. forces arising from system constraints, allows for a consistent assessment of *ex-ante* and *ex-post* dynamics of the dynamical system. There are several advantages to this modeling framework, as it allows to describe multiple equilibria, out-of-equilibrium dynamics as well as prisoner's dilemma situations, and model behavioral assumptions different from rationality and utility maximization. Also slow price-adaptation processes and the monetary dynamics of financial stocks and flows may be included.

We illustrate some of these benefits by providing a dynamic version of a static exchange textbook model: We replace optimization with a process described by economic forces exercised by the agents to improve (but not perfectly optimize) their situation, and change the price-adaptation process from instantaneous to continuous, and study the convergence to the stationary state (section 4). In section 5 we discuss findings, caveats and potential applications of the approach. Section 6 concludes.

2 General equilibrium theory was inspired by physics

Newton's 'Principia' (1687) not only initiated a scientific revolution in physics, but also had a substantial impact on other fields of study. While Newton is (disputably) cited that he "could calculate the motions of erratic bodies, but not the madness of a multitude" (Francis, 1850, p. 142), modern economics, starting with endeavored to imitate his methodology of the natural sciences. "Newton's success in discovering the natural laws of motion" (summarized in section 3) inspired the search for "general laws of economics" (Hetherington, 1983, p. 497). The "Newtonian method" of deducing several phenomena from certain primary principles (Redman, 1993, pp. 211–5) was applied by Adam Smith (1759; 1776), "first to ethics and then to economics" (Blaug, 1992, p. 57).

Mirowski (1989, pp. 26–30) summarized that the founding fathers of neoclassical economics followed the "Laplacian dream" of finding "the single mathematical formula that described the entire world" with "rigid determinism". Walras (1874, p. 71) motivated his use of mathematics with the analogy that the pure theory of economics is "a physico-mathematical science like mechanics", and argued in 1909 that the way economics proceeds is rigorously identical to the one of rational and celestial mechanics (Walras, 1960, p. 5)[1]. Similarly, Pareto (1896, p. iii) tried to sketch economics as a "natural science" in analogy to mechanics (McLure and Samuels, 2001), explicitly linking pure (and applied) economics to pure (and applied) mechanics (Pareto, 1907, p. 146). I. Fisher (1892, pp. 85–6) and Pareto (1897, pp. 12–3) even stated the influence of physics for their theories explicitly by offering concordance tables of mechanics and economics. As an example, they relate marginal utility to a "force" f_i^j of a conservative field, generated by the gradient of the utility function (I. Fisher, 1892, pp. 85–6; Pareto, 1896, pp. 35–6):

$$f_i^j = \frac{\partial U^j(x^j)}{\partial x_i^j}. \tag{1}$$

General equilibrium theory postulates that there exists a set of prices that will lead to an

[1] "sa manière de procéder est rigoureusement identique à celle de deux sciences physico-mathématiques des plus avancées et des plus incontestées: la *mécanique rationelle* et la *mécanique céleste*" (Walras, 1960, p. 5).

overall 'Walrasian' equilibrium in a whole economy with several connected markets (Arrow and Hahn, 1983; Walras, 1874). A market equilibrium is a situation where "each agent in the economy is doing as well as he can given the actions of all other agents", and maximization subject to constraints is "common to nearly all economic models" (Mas-Colell et al., 1995, pp. 307–314), even identified as a defining principle of economics itself (Dixit, 1990, p. 1). Households maximize their utility, and firms their profits (i. e. revenue minus costs, which corresponds to firms' utility), anticipating the reactions of other market participants, as introduced by Cournot (1838, 1897). This combination of "Cournot's Newtonian calculus method of maximizing with Walras's equations of general equilibrium" resulted in economists having "Newtonian Paradise Regained", as Samuelson (1983, pp. xvii–xviii) argued in his influential book 'Foundations'. In 1907, Pareto was aware that the "theory of statics" of determining the equilibrium in analogy to mechanics is "most advanced", while "nothing is known about dynamic theory" (Pareto, 1971, pp. 104–5). He recognized that the "theory must be extended to encompass dynamics", but had "limited success" in performing this task (Donzelli, 1997; McLure and Samuels, 2001, p. 37).

In today's dynamic equilibrium models, the static optimization is replaced by a dynamic problem of optimal control. Similar to the calculation of motion in mechanics (Janová, 2011), variational calculus is used to maximize (or minimize) a specific Lagrangian function. The Euler–Lagrange equation describes the inter-temporal trade-off. As long as the dynamics in such models are restricted to the description of a unique optimal stable path in equilibrium, they resemble quasi-static processes in physics "in which the system is ... at equilibrium at every point between its initial and final states" (Berry et al., 1978, p. 126).

For the "early neoclassicals", however, dynamics "did *not* mean *intertemporal* choices or equilibria but instead the adaptive *processes* that were thought to converge on the states analyzed in *static* theory" (Leijonhufvud, 2006, p. 29, emphasis in original). As an example, Pareto (1897, 1907) wanted to explain the path towards equilibrium, as summarized by Pikler (1955, p. 305):

> "Every individual strives to move according to his desires ('goûts') and to change

> existing configurations in the direction of these desires. However, every individual in his virtual moves is subject to certain external constraints ('obstacles') imposed on him by society. Actually he will be able to satisfy his desires along certain open paths ('sentiers') only, and when he reaches a certain terminal point ('point terminal'), namely that of maximum attainable satisfaction, he does not wish to move further. The individual's behavior is then in equilibrium."

Leijonhufvud (2006, p. 30) argues that "very little has been done to address the unfinished business of the older neoclassical theory."

In the following, we try to contribute to this 'unfinished business' and show how analogies to classical mechanics can help to provide a sound basis to study out-of-equilibrium dynamics and interaction. Picking up the challenge by Pareto to develop a framework in which each agent seeks to change the existing configuration in the direction of his desires but is subject to external constraints, we try to revisit the foundations of economic models by developing a dynamic modeling framework inspired by mechanics. If this approach proves successful, 'General Constrained Dynamic' (GCD) models could be a fitting denomination. This perspective on the convergence towards equilibrium can be useful for theories where economic agents cannot anticipate all constraints correctly, or are unable to 'think at the margins' and jump to the highest point reachable on the utility hill. Another application are markets that fail to determine the prices such that a general market equilibrium is reached, or if multiple such equilibria exist. We present our modeling approach and discuss the relation to these theories subsequently.

3 General constrained dynamics – extending analogies between economics and mechanics

3.1 Lagrangian constrained dynamics

The framework is inspired by the description of interacting 'bodies' under constraint in mechanics and the bases of our approach are the concepts of *economic forces* and *economic power*, which build on the concepts of physical forces and the reciprocal value of mass. In physics, one advantage of Lagrangian mechanics (Lagrange, 1788) is the description of motion under constraint. To make the remarkable similarity of our economic framework to the laws of mechanics obvious, let us recall how multiple forces f_i^j influence location coordinates s_i and velocity coordinates v_i of a mass point with constant mass M, assuming the forces depend only on the vectors s and v:

$$\dot{s}_i(t) = v_i(t), \tag{2}$$

$$\dot{v}_i(t) = \frac{1}{M} \sum_{j=1}^{J} f_i^j(s, v). \tag{3}$$

$1/M$ is the factor that determines the influence of a certain force f on the velocity v of a mass point. f/M corresponds to an acceleration or deceleration of a mass point.

Additionally, physical constraints $C^k(s, v)$ may restrict the dynamics of the system. While forces are mostly defined directly by direction and magnitude, constraint forces adapt via Lagrange multipliers λ^k such that the restriction of motion will always be satisfied. This results in additional 'constraints forces' c_i^k with the equations of motion given by:

$$\dot{s}_i(t) = v_i(t), \tag{4}$$

$$\dot{v}_i(t) = \frac{1}{M} \sum_{j=1}^{J} f_i^j(s, v) + \sum_{k=1}^{K} c_i^k(s, v), \tag{5}$$

$$0 = C^k(s, v). \tag{6}$$

In general, the following holds for the constraint forces c_i^k (Flannery, 2011): If a constraint C^k depends not on v_i (thus $\partial C^k / \partial v_i \equiv 0$), it is called 'holonomic' in s_i. Then, the constraint force $c_i^k(s, v)$ is given by:

$$c_i^k(s, v) = \lambda^k \frac{\partial C^k}{\partial s_i}. \tag{7}$$

If C^k is 'non-holonomic' in s_i because it depends on v_i, the constraint force $c_i^k(s, v)$ is given by:

$$c_i^k(s,v) = \lambda^k \frac{\partial C^k}{\partial v_i}. \tag{8}$$

Eq. (4–8) build a system of differential-algebraic equations which can be solved numerically for $s(t)$ and $v(t)$.[2] We transfer these concepts to economics.

3.2 An economic modeling framework

In our economic framework, the model economy is described by J agents and I variables $x_i(t)$. An economic state at time t is described by the value of the variables that depend on the aim of the model. In general, they may correspond to any stocks or flows of commodities, resources, financial liabilities, or any other variables or parameters that are relevant for the economy such as prices or interest rates. In correspondence to the mechanical laws, we assume that the dynamics of the economic model are the result of agents wanting to change the state of the model economy, represented by different forces f_i^j.

$$\dot{x}_i = \sum_{j=1}^{J} \mu_i^j f_i^j(x). \tag{9}$$

The μ_i^j correspond in some sense to the inverse of the mass in Newtonian theory, but in contrast to mass they depend both on agents j and variables i. We call them 'power factors' because they represent the ability of a specific force f_i^j to change the state of the economy, in line with Russell (2004, p. 23) who defined power "as the production of intended effects." Note that this factor relates closely to what Pareto called "force" in a social context: "the capacity to influence an economic or social situation" (McLure and Samuels, 2001, p. 59). The total impact on the variable x_i is the product of economic force and power $\mu_i^j f_i^j$, i. e. the product of *desire* and *ability* of agent j to influence an economic variable x_i. If power factors are only dependent on variables i but not on agents j, they may also be interpreted as determining

[2] Flannery (2011) deduced that if the constraint C^k depends on $\dot{v}(t)$, c_i^k is given by $\lambda^k \cdot \partial C^k / \partial \dot{v}(t)$.

the adjustment speed of certain variables as in new-Keynesian models. Note that a stationary state in this system is defined if these accelerations equilibrate, i. e. $\dot{x}_i = 0$ for all i.

3.3 Bounded rationality as gradient climbing

In many economic models, the forces f_i^j may be described as the gradient of a utility function U^j depending on the variable x_i. If an increase of a certain variable x_i leads to a high increase in utility U^j, the agents try to augment this variable over time by applying positive forces f_i^j. This revives the idea of Pareto and Fisher, considering marginal utility as force.[3] These economic forces with corresponding power factors constitute a special case of Eq. (9), formalizing an adaptive, "procedural rationality" of "gradient climbing" (Leijonhufvud, 2006, p. 31):

$$\dot{x}_i = \sum_{j=1}^{J} \left(\mu_i^j \frac{\partial U^j(x)}{\partial x_i} \right), \tag{10}$$

$$0 = Z^k(x, \dot{x}). \tag{11}$$

Forces that can be described as gradients of utility functions are 'bounded rational' in the economic sense, as such behavior is congruent with trying to 'climb up the utility hill'. According to the Helmholtz decomposition (Helmholtz, 1858; Stokes, 1849), any vector field and thereby any economic force can be decomposed into an irrotational (conservative) and a rotational (solenoidal) field. 'Bounded rationality' in economics corresponds to 'conservative forces' in physics that are gradients of some potential. In the same sense as irrotational fields represent 'rational' gradient-following behavior, rotational forces represent 'irrational' behavior, as they are tantamount to trying to 'walk in circles around the utility hill'.

[3] A minor side effect is the consequence of our treatment of 'marginal utility'. Originally, utility was used to compare different preferences, but the absolute value of utility was without meaning. Thus any strictly monotonic transformation from U to U' did not change the model results, as prices and quantities are determined by fractions such as $\frac{\partial U}{\partial x_i} / \frac{\partial U}{\partial x_j}$, where the transforming function cancels out (E. Smith and Foley, 2008, p. 11). In our model, this transformation influences the magnitude of the forces and leads to differences in the dynamics.

3.4 Constraint forces in economic models

An analogy to mechanical constraints C^k exists in economic models: They consist of "definitions or identities" specifying relations between variables "that hold by definition" (Allen, 1982, p. 4) such as the national income account identity. Other constraints include specific model assumptions, such as conservation laws of goods under exchange, or production functions. They can be written typically as:

$$0 = C^k(x, \dot{x}). \tag{12}$$

In both physics and economics, constraints are additional conditions which variables have to fulfill. Constraints in physics add constraint forces to the equations of motion. We transfer this concept to economics.

As an example, consider a pure exchange economy with two agents, thus the amount of goods is conserved. Both may wish to increase their inventory of a certain good, so they try to increase their stock, which leads to demand without supply, or excess demand. This would correspond to the *ex-ante* state. But the conservation law that acts as a constraint has to guarantee that this does not lead to the generation of goods out of nothing. These constraints result in *ex-post* dynamics different from the dynamics targeted by individual decisions. The constraint forces can be modeled in analogy to classical mechanics by Lagrange multipliers λ^k and the gradient of C^k. This guarantees that identities hold without the need to explicitly define *a priori* which variables determine others. Altogether, forces f applied by all agents and constraint forces c create the *ex-post* dynamics:

$$\dot{x}_i = \sum_{j=1}^{J} \mu_i^j f_i^j(x) + \sum_{k=1}^{K} c_i^k(x, \dot{x}), \tag{13}$$

$$0 = C^k(x, \dot{x}). \tag{14}$$

Analogously to the constraint forces in physics, c_i^k can be calculated as

$$c_i^k(x, \dot{x}) = \lambda^k \frac{\partial C^k}{\partial x_i} \tag{15}$$

if $\partial C^k / \partial v_i \equiv 0$ (holonomic constraints) and as

$$c_i^k(x, \dot{x}) = \lambda^k \frac{\partial C^k}{\partial \dot{x}_i} \tag{16}$$

for non-holonomic constraints that depend on $\dot{x}_i$.[4] Again, Eq. (13–16) build a system of differential-algebraic equations which can be solved numerically for $x(t)$ and $\dot{x}(t)$.

This way, economic models with stock-flow relations such as (net) investment I being the change of the capital stock K can be implemented using the constraint $0 = \dot{K} - I$. If the utility function depends not only on x_i, but also on its derivative $\dot{x}_i$, $\dot{x}_i$ has to be renamed to a variable x_j and the two have to be joined by an additional constraint $C = \dot{x}_i - x_j = 0$ which is non-holonomic with respect to x_i and holonomic with respect to x_j. Thus one has to add the constraint force c_i given by $\lambda \frac{\partial C}{\partial \dot{x}_i} = \lambda$ to the time evolution of $\dot{x}_i$, while c_j given by $\lambda \frac{\partial C}{\partial x_j} = -\lambda$ has to be added to the time evolution of $\dot{x}_j$.

3.5 Advantages of the approach

We suggest that this modeling approach has numerous advantages, further elaborated in the discussion, as it is designed to be able to:

(1) incorporate behavioral assumptions different from rationality and utility maximization, without the need for restrictive macroscopic assumptions about individual behavior to permit aggregation, allowing for non-optimal outcomes of individual strategies (discussed in section 5.1),

(2) allow for a formal distinction and explicit modeling of *ex-ante* and *ex-post* dynamics of

[4] In general, constraints may also depend on $\int x_i$ or higher derivatives of x_i. Extending the analogy to classical mechanics, c_i^k in general is given by the partial derivative of C^k with respect to the highest time derivative of x_i it depends on, multiplied by λ^k.

the system, where unanticipated constraints lead to constraint forces (discussed in section 5.2),

(3) discuss slow price adaptation and out-of-equilibrium dynamics (discussed in section 5.3),

(4) treat stocks and flows and their constraints and can thus include multiple financial assets consistently (discussed in section 5.4),

(5) explicitly formalize economic power as a parameter describing the ability of an agent to influence certain variables (discussed in section 5.5),

(6) include some well-known general equilibrium solutions as fixed points of the dynamical system (discussed in section 5.6).

4 A dynamic version of a static Edgeworth Box exchange model

In this section, we apply our novel modeling framework to a simple exchange model where well-known equilibrium solutions exist, and study different assumptions about the speed of price adaptations and trade. In the standard Edgeworth box exchange model, "the ways in which economic power pulls and tilts visible economic variables remain invisibly obscure" (Bhaduri, 2016, p. 31). We address this issue by explicitly incorporating economic power in our framework and show that the dynamics of the model depend on the assumptions about the auctioneer's power to influence the price. Assumptions from full power to zero power are possible. Price and quantity adaptation occurs while exchange takes place and agents try to push the economy towards their highest individual improvement rather than optimizing 'globally'. We show that the model converges to the usual equilibrium for specific power parameters, but that also other stationary states can be reached. We hope that this example offers both additional insights into a standard model and provide an intuition of our modeling approach.

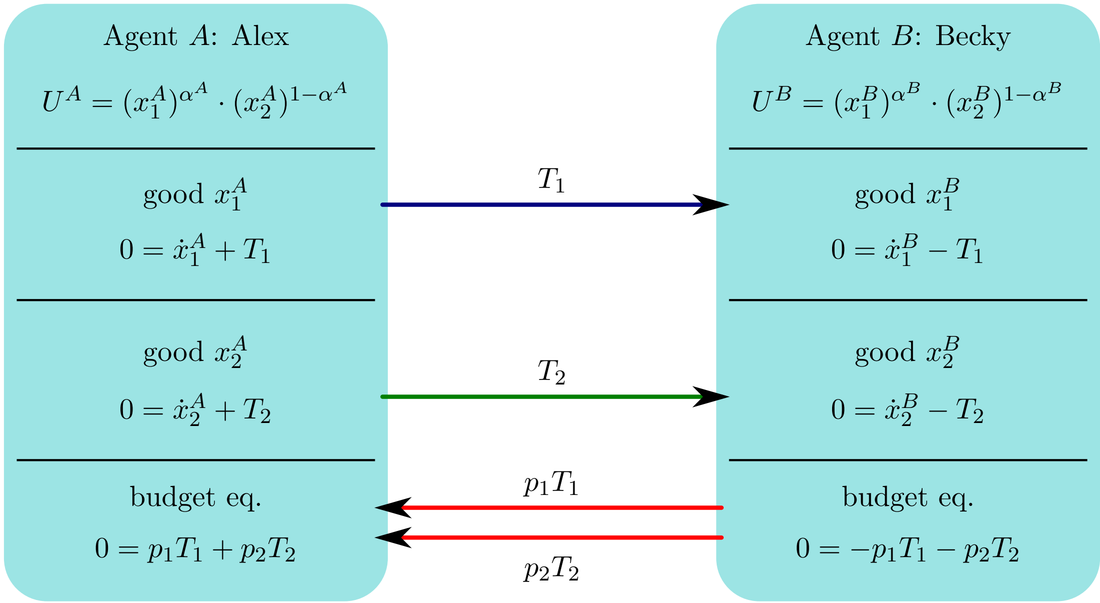

Figure 1: Structure of simple exchange model. Two agents A and B with utility functions U^A and U^B own stocks x_1^A, x_2^A, x_1^B, and x_2^B of good x_1 with price p_1 and x_2 with price p_2. T_i corresponds to a transfer of good i from A to B.

4.1 Description of the exchange model

We now exemplify our approach using a simple exchange model with two agents (Alex and Becky) and two commodities x_1 and x_2. x_i^j is the stock of the commodity currently owned by j ($j \in \{A, B\}$). The total amount of x_i is $x_i^{tot} = x_i^A + x_i^B$. p_i is the price of x_i. Without any loss of generality, x_1 is the numeraire, thus $p_1 = 1$. For every agent j, a 'selfish' utility function $U^j(x_1^j, x_2^j)$ fulfilling the Inada conditions[5] maps all possible commodity bundles (x_1^j, x_2^j) into the real numbers $\mathbb{R}$. We treat x_i^j, and p_2 as five time-dependent variables, and their change with time is given by $\dot{x}_i^j$ and $\dot{p}_2$.

In the conventional static model, the equilibrium price p_2^* and the final allocation of goods can be determined by maximizing a Lagrangian function subject to constraints. The implicit assumption is that prices adjust to an equilibrium price p_2^* (e. g. through an auctioneer) where demand equals supply for all goods *before* goods and money are exchanged. In the following, we

[5] U^j is strictly increasing, strictly concave, continuously differentiable and $U'(0) = \infty$ and $U'(\infty) = 0$ in every argument.

present a GCD model as alternative formulation where price adaptation occurs *while* exchange takes place, and where agents do not optimize 'globally' but individually 'push' the economy in the direction of their highest 'local' improvement.

The structure of the model is represented in figure 1. The first two constraints are the conditions of "pure exchange" (F. M. Fisher, 1983, p. 54):

$$D_1 = \dot{x}_1^A + \dot{x}_1^B = 0 \quad (= \dot{x}_1^A + T_1 = \dot{x}_1^B - T_1), \tag{17}$$

$$D_2 = \dot{x}_2^A + \dot{x}_2^B = 0 \quad (= \dot{x}_2^A + T_2 = \dot{x}_2^B - T_2). \tag{18}$$

Every increase in good x_1^j must be compensated by a proportional decrease in x_2^j, leading to the budget constraints:

$$D^j = p_1 \dot{x}_1^j + p_2 \dot{x}_2^j = 0. \tag{19}$$

This equation is called "condition of barter" by Negishi (1962, p. 659), while F. M. Fisher (1983, p. 54) prefers "no swindling", meaning that "no agent will ever give up something unless what he gets for it is of equal value".[6] Because Eq. (17) is redundant to the other three, we use the three constraints D_2, D^A, and D^B in our model and accordingly three Lagrangian multipliers λ_2, λ^A, and λ^B that will show up in the differential equations, guaranteeing that these constraints will always be fulfilled.

Applying Eq. (18) and (19) to the general constrained dynamic model presented in section 3 with $\mu_1^j = \mu_2^j = \mu^j$, the time evolution yields:

$$\dot{x}_1^j = \mu^j \frac{\partial U^j}{\partial x_1^j} + p_1 \lambda^j, \tag{20}$$

$$\dot{x}_2^j = \mu^j \frac{\partial U^j}{\partial x_2^j} + p_2 \lambda^j + \lambda_2. \tag{21}$$

[6] Note that the conventional budget constraint $p_1 \left[x_1^j(T) - x_1^j(0)\right] + p_2 \left[x_2^j(T) - x_2^j(0)\right] = 0$ can be obtained from Eq. (19) by integration only if the relative price p_1/p_2 is constant during the exchange process and path dependence of trading at different prices is excluded.

The Lagrangian multiplier λ_2 corresponds to the constraint D_2, and λ^j correspond to constraints D^j that limit the choice of each agent individually (because agents cannot swindle and have to buy the commodity for money).

The targeted change in goods (i. e. the change of goods the agents are targeting without respecting the conservation constraint of the total stock of good x_2) can be calculated if the exchange constraint D_2 is first neglected (i. e. for $\lambda_2 = 0$). If then the λ^j are eliminated, one can calculate the targeted change in goods as:

$$\dot{x}_2^{j\top} = \frac{\mu^j}{1+(p_2)^2}\left(\frac{\partial U^j}{\partial x_2^j} - p_2\frac{\partial U^j}{\partial x_1^j}\right). \tag{22}$$

Summing over $\dot{x}_2^{j\top}$ yields z_2 which is excess demand for good x_2.

$$z_2 = \sum_{j=A,B} \dot{x}_2^{j\top}. \tag{23}$$

To model the price evolution $\dot{p}_2$, we introduce an auctioneer that adapts the price slowly while trade takes place.

4.2 The auctioneer

Usually, it is assumed that the auctioneer fixes the price before any trade happens (Negishi, 1989). If the assumption that agents optimize perfectly is dropped, the auctioneer has to have a different rule than in its traditional formulation. We therefore integrate an explicit auctioneer's process following the description by Arrow and Hahn (1983). Their approach relies on the excess demand function $\boldsymbol{z}(\boldsymbol{p})$ which is determined by calculating the total demand for every good minus supply at a given price. Then they propose the following tatonnement process:

Let $G_i(z_i)$ be a sign-preserving function of z_i, with $G_i(0) = 0$, and differentiable with respect to z_i and $G_i' > 0$. Then for all i:

$$\dot{p}_i = 0 \text{ if } p_i \leq 0 \text{ and } z_i(p) < 0 \tag{24}$$

$$\dot{p}_i = G_i \left(z_i p_i \right) \text{ otherwise.} \tag{25}$$

If this is true, then "(a) prices change if and only if the economy is not in equilibrium; (b) for all finite t, $\boldsymbol{p}(t) > \mathbf{0}$ if $\boldsymbol{p}(0) > \mathbf{0}$. The 'error' that causes a given price to change is a disparity between the planned transactions of agents in the market in which that price is called" (Arrow and Hahn, 1983, pp. 266–7).

As agents in our model do not optimize their trade based on a complete knowledge of their utility function but rather try to follow the gradient, it seems reasonable to replace excess demand as Arrow and Hahn (1983) use it by excess demand following equation (23), which changes with time. Thus if for good x_2 the sum of these targets z_2 is bigger than 0, we have current excess demand, and the tatonnement process increases the price of good x_2. The price evolution $\dot{p}_2$ in the model according to section 3 is then determined by the auctioneer's force given by equation (25) with a power factor of μ^{auc} and the constraint forces on p_2:

$$\dot{p}_2 = \mu^{auc} G_2 \left(\frac{1}{1 + (p_2)^2} \sum_{j=A,B} \mu^j \left(\frac{\partial U^j}{\partial x_2^j} - p_2 \frac{\partial U^j}{\partial x_1^j} \right) \right) + \sum_{j=A,B} \lambda^j x_2^j. \tag{26}$$

This explicates the power of the auctioneer as suggested by Bhaduri (2016). Note that the force applied by the auctioneer is one of the 'general' forces of Eq. (13) not derived from any utility function. With $\mu^{auc} = 0$, the auctioneer is absent in the model. The full set of first order partial differential equations is:

$$\dot{x}_1^j = \mu^j \frac{\partial U^j}{\partial x_1^j} + \lambda^j, \tag{27}$$

$$\dot{x}_2^j = \mu^j \frac{\partial U^j}{\partial x_2^j} + \lambda_2 + p_2 \lambda^j, \tag{28}$$

$$\dot{p}_2 = \mu^{auc} G_2 \left(\frac{1}{1 + (p_2)^2} \sum_{j=A,B} \mu^j \left(\frac{\partial U^j}{\partial x_2^j} - p_2 \frac{\partial U^j}{\partial x_1^j} \right) \right) + \sum_{j=A,B} \lambda^j x_2^j, \tag{29}$$

$$0 = D_2 = \dot{x}_2^A + \dot{x}_2^B, \tag{30}$$

$$0 = D^j = \dot{x}_1^j + p_2 \dot{x}_2^j. \tag{31}$$

Eliminating the Lagrangian multipliers, with δ_{ij} being the Kronecker delta, analytically one gets:

$$\dot{x}_1^j = -p_2 \dot{x}_2^j, \tag{32}$$

$$\dot{x}_2^j = \frac{1}{1+(p_2)^2} \sum_j (\delta_{ij} - 0.5) \left[\mu^j \left(\frac{\partial U^j}{\partial x_2^j} - p_2 \frac{\partial U^j}{\partial x_1^j} \right) \right], \tag{33}$$

$$\dot{p}_2 = \mu^{auc} G_2 \left(\frac{1}{1+(p_2)^2} \sum_j \mu^j \left(\frac{\partial U^j}{\partial x_2^j} - p_2 \frac{\partial U^j}{\partial x_1^j} \right) \right) - \sum_j \left[\dot{x}_2^j \left(p_2 \dot{x}_2^j + \mu^j \frac{\partial U^j}{\partial x_1^j} \right) \right]. \tag{34}$$

Both systems of equations (27–31 resp. 32–34) can be solved numerically. $\dot{x}_1^j$ and $\dot{x}_2^j$ describe the exchange process, driven by the forces to increase utility and constraint forces guaranteeing the exchange and no-swindling condition. $\dot{p}_2$ specifies the price adaptation, influenced by the auctioneer and constraint forces.

4.3 Stationary states of the model

Eq. (32–34) allow for some more insights into the model. If an equilibrium is reached, Eq. (32) is trivially fulfilled. Eq. (33) can be rewritten using $\dot{x}_i^j = 0$ such that

$$\mu^A \left(\frac{\partial U^A}{\partial x_2^A} - p_2 \frac{\partial U^A}{\partial x_1^A} \right) = \mu^B \left(\frac{\partial U^B}{\partial x_2^B} - p_2 \frac{\partial U^B}{\partial x_1^B} \right). \tag{35}$$

For $\mu^{auc} = 0$, Eq. (34) is trivially fulfilled, while for $\mu^{auc} > 0$, the following condition has to hold:

$$\mu^A \left(\frac{\partial U^A}{\partial x_2^A} - p_2 \frac{\partial U^A}{\partial x_1^A} \right) = -\mu^B \left(\frac{\partial U^B}{\partial x_2^B} - p_2 \frac{\partial U^B}{\partial x_1^B} \right). \tag{36}$$

For every agent j, either one of the first two cases, or alternatively (for $\mu^{auc} = 0$) the third has to be fulfilled:

1. $\mu^j = 0$: If the agents have no power and are thus unable to exert a certain influence on their stocks, their utility functions do not matter as intended.

2. $\frac{\partial U^j}{\partial x_2^j} = p_2 \frac{\partial U^j}{\partial x_1^j}$. The latter is well known as the first order equilibrium condition for consumers in general equilibrium models (Arrow and Hahn, 1983). The ratio of prices equals the ratio of marginal utilities, thus the utility from the last monetary unit spent on each good must be the same[7].

3. If $\mu^{auc} = 0$, another case is possible: $\mu^A \left(\frac{\partial U^A}{\partial x_2^A} - p_2 \frac{\partial U^B}{\partial x_1^A} \right) = \mu^B \left(\frac{\partial U^B}{\partial x_2^B} - p_2 \frac{\partial U^B}{\partial x_1^B} \right) \neq 0$. This is an interesting and non-trivial case and corresponds to the result that is attained in figure 2. It is a generalization of Gossen's law and indicates that trade leads to a situation where the forces cancel each other out at current prices, even though there would be a Pareto improving allocation on the contract curve. This allocation is not reached because in this case, it is assumed that agents are not able to adapt the prices.

If all power factors are non-zero, all stationary states fulfill the equilibrium conditions of the static model, and all equilibria of the static model are stationary states of the dynamic version. The sets of rest points of the systems are therefore identical.

4.4 Numerical dynamics

We solve the differential equations numerically and the results are plotted in figure 2. We can see that in absence of an auctioneer (left plot), the point of convergence does not lie on the contract curve, but constitutes an example of case 3 where $\mu^j \left(\frac{\partial U^j}{\partial x_2^j} - p_2 \frac{\partial U^j}{\partial x_1^j} \right)$ has a constant value $\neq 0$ for all j. If we switch on the auctioneer by setting $\mu^{auc} = 0.1$ (middle plot), the equilibrium allocation lies on the contract curve as expected, but it is not the final allocation that would have been calculated based on utility functions and initial endowments (right plot with 'almighty' auctioneer). The process we obtain in the middle is described fairly accurately by F. M. Fisher (1983, pp. 14–6):

> "In a real economy, however, trading, as well as production and consumption,

[7] As only relative prices matter in the economic process, one can easily set $p_i = \frac{\partial U^j}{\partial x_i^j}$ which explains why Hands (1993) and Mirowski (1989) think that prices are given by gradients of some potential. This may be the case in equilibrium, but is not valid out of equilibrium, which makes the discussion of general conservation laws arising from this identity obsolete, because path dependency indeed has to be considered.

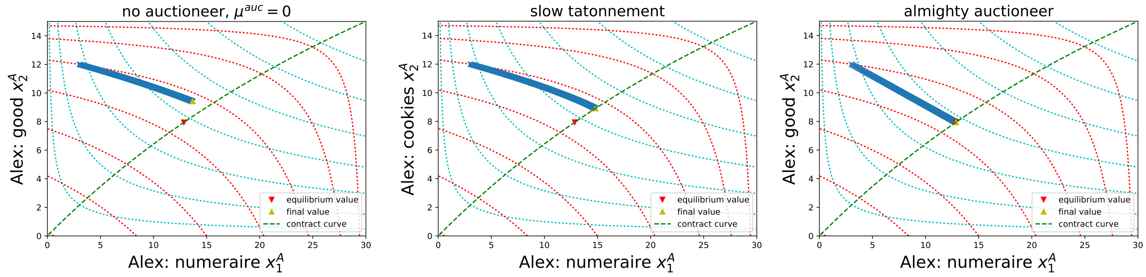

Figure 2: Edgeworth boxes of a simple exchange model with two commodities (numeraire + good) and two agents, Alex and Becky. The lower left (resp. upper right) corner corresponds to an endowment of (0,0) for agent A (resp. B). Starting from the particular origin, the dotted cyan (red) lines correspond to equal utility depending on the stocks of agent A (agent B), with $U^A = (x_1^A)^{0.4}(x_2^A)^{0.6}$, $U^B = (x_1^B)^{0.5}(x_2^B)^{0.5}$. The total size of the box corresponds to the conserved quantity of the stocks. The blue crosses show the dynamics of the stocks in the Edgeworth Box. The gray triangle corresponds to the last value of the time evolution. The green dashed contract curve indicates all the Pareto optima, while the red triangle indicates the 'conventional' equilibrium value, obtained with an equilibrium price $p_2^* \approx 2.46$, calculated using the textbook version of the model.
The plots use $2p_2^*$ as a starting price, and the current price is visible as the inverse of the slope of the time evolution. The left plot shows the trading process without auctioneer ($\mu^{auc} = 0$), while the plot in the middle uses the tatonnement process of Eq. (26) with $\mu^{auc} = 0.1$ and G_2 the identity function, and the right plot assumes that the 'almighty' auctioneer knows the equilibrium price p_2^* and has full control $\mu^{auc} = \infty$ over the price. The power parameters are $\mu^A = 1$, $\mu^B = 2.5$, while the initial endowments at $t = 0$ are $x_1^A = 3$, $x_2^A = 12$, $x_1^B = 27$, $x_2^B = 3$.

goes on out of equilibrium. It follows that, in the course of convergence to equilibrium (assuming that occurs), endowments change. In turn this changes the set of equilibria. Put more succinctly, the set of equilibria is path dependent – it depends not merely on the initial state but on the dynamic adjustment process."

If the auctioneer knows the price and has perfect control, we obtain convergence to the 'standard' solution. If the power factor μ^{auc}, which corresponds to the adjustment speed of the price, is finite, the final allocation will diverge from the standard equilibrium value.

5 Discussion

General constrained dynamic models constitute a novel framework to dynamically model economic problems in analogy to Lagrangian mechanics, and provide an alternative to the standard general equilibrium framework.

We argued in section 3 that our approach is suited to (1) incorporate behavioral assumptions different from optimization, without the need for aggregation to solve the model, (2) distinguish between *ex-ante* and *ex-post* dynamics, (3) describe slow price adaptation and out-of-equilibrium dynamics, (4) consistently treat monetary assets, (5) explicitly formalize economic power, and (6) include well-known general equilibrium solutions as fixed points of a dynamical system. In the following, we discuss these six claims with reference to the models presented and reflect on potential caveats.

5.1 Behavioral assumptions and non-optimal outcomes

Most equilibrium theories are "zealous in insisting that all decisions" are "consistent with maximizing behavior" and perfect rationality (Akerlof, 2002, pp. 411–2). However, psychological research shows that bounded rationality prevails (Kahneman, 2003, 2011; Simon, 1955). Akerlof argues "that reciprocity, fairness, identity, money illusion, loss aversion, herding, and procrastination help explain the significant departures of real-world economies from the competitive, general-equilibrium model." He concludes "that macroeconomics must be based on such behavioral considerations" (Akerlof, 2002, pp. 427–8).

For a long time, neoclassical economists tried to aggregate a society of utility maximizers into a single representative individual. The economy as a whole could then be described by a single concave social welfare function, which ensures that a unique and stable equilibrium exists. Unfortunately, "the hypothesis of individual rationality, and the other assumptions made at the micro level" are "not enough to talk about social regularities" (Rizvi, 1994, p. 363). Debreu (1974), Mantel (1974), and Sonnenschein (1972) demonstrated that given some heterogeneity in preferences or endowment among agents, multiple equilibria may exist.

Not even identical preferences are sufficient to guarantee a unique equilibrium (Kirman, 1992; Kirman and Koch, 1986), but it is necessary that "macro-level assumptions ... restrict the *distribution* of preferences or endowments" (Rizvi, 1994, p. 359). Aggregation is possible if and only if demand is independent of the distribution of income among the agents (Gorman, 1961; Stoker, 1993), which Rizvi (1994, p. 363) calls an "extremely special situation". Blundell and Stoker (2005, p. 350) argue that "convenient constructs such as a 'representative agent' have, in fact, no general justification", but state a "practice of ignoring or closeting aggregation problems" (p. 385), for example by postulating that individual demand functions are linear in income. Also the situation of a prisoner's dilemma, known from game theory (Nash, 1951), where individually optimal behavior can lead to stable equilibria which constitute the worst scenario for the totality of players, is disregarded. If models are formulated such that a unique and stable equilibrium exists, it is not necessary to determine whether, why, and how the model economy settles at a specific equilibrium (path). As a result, one can either disregard the out-of-equilibrium dynamics, as is done in many models (F. M. Fisher, 1983, 2011), or study them with perturbation theory, using "Taylor series approximations to the solution of a DSGE model around its deterministic steady state" (Fernández-Villaverde et al., 2016, p. 13).

In contrast, general constrained dynamic models allow for diverse specifications of utility functions or completely different behavioral assumptions without causing fundamental problems for the mathematical approach. In the dynamic version of the text-book exchange model, we replaced 'global' rationality and optimization through a 'local' process described by economic forces exercised by the agents to improve (but not perfectly optimize) their situation. Also differential games such as prisoner's dilemmata may be integrated into the framework (Glötzl, 2016). Nevertheless, the conventional equilibrium solutions may still be explicitly present as rest points of the dynamical system.

The problem of aggregation and rational behavioral has been addressed by agent-based models. ABM numerically simulate the actions and interactions of a finite number of autonomous agents. They can implement locality and search costs, bounded rationality and heterogeneity among consumers and firms, the possibility of coordination failures, defaults and network effects (Ballot et al., 2014; Battiston et al., 2007; Feng et al., 2012). Usually, these approaches are based on discrete time frameworks and include a distinct sequence of events within each period.

Our modelling framework approaches these problems differently. While ABM strongly depart from the standard setup of economic models and reach a very high degree of complexity, general constrained dynamic models may still be formulated based on utility functions and constraints known from general equilibrium models. They thus allow to relax the restrictions about rationality and aggregation known from general equilibrium models, while remaining simpler and more easily accessible than most ABM. At the same time, GCD models may also be of use to economists working on such agent-based models. Usually, the aggregated results of ABM are compared to DSGE models (Fagiolo and Roventini, 2012), but as multiple equilibria and instabilities are not found in DSGE models, a dynamic model of constrained dynamics that is able to capture these may be more suitable for this type of meta modeling.

5.2 Distinction between ex-ante and ex-post dynamics

Perfect rationality implies that the constraints imposed by other agents or system properties are fully anticipated. Departing from this assumption requires to differentiate between *ex ante* (action planned) and *ex post* (actual action) (Myrdal, 1939). The setup of our model framework allows for a clear distinction between *ex-ante* dynamics that describe how the economy would develop without constraints and *ex-post* dynamics that describe the actual development. In the example, this was shown by the fact that constraints that were not incorporated in the agents' decision process generate constraint forces which account for the difference between the planned (ex-ante) and the actual (ex-post) dynamics.

5.3 Slow price adaptation and out-of-equilibrium dynamics

The trading process in equilibrium models is usually described such that economic agents 'wait' until prices are set (by the hypothetical auctioneer): "[N]o actual transactions, and therefore no production and consumption activities, take place at disequilibria when prices are changed" (Negishi, 1989, p. 281), thus a distinct sequence is assumed. One could say that price adaptation is infinitely faster than trade. We cited F. M. Fisher (1983, pp. 14–6) in section 4.4 arguing that price adaptation may be slow(er) and the set of equilibria path dependent, which "makes the calculation of equilibria corresponding to the initial state of the system essentially irrelevant." Therefore, he argues that "it is not at all obvious that the actual economy is stable" and "we largely lack a convincing theory of why that should be so" (p. 35).

A similar argument goes back to Keynes (1936), who "denies that orthodox equilibrium analysis provides an adequate account of disequilibrium phenomena" (Clower, 1965, p. 276): "the key to the Keynesian theory of income determination is the assumption that the vector of prices, wages, and interest rates does not move instantaneously from one full employment equilibrium position to another" (Barro and Grossman, 1971, p. 82). Clower (1965), Leijonhufvud (1968), and Patinkin (1965) considered unemployment to be a relevant disequilibrium phenomenon caused by "wage stickiness" such that prices adjust slowly, so there exists "false trading" out of equilibrium (De Vroey, 2004, pp. 115–46). These economists in turn pursued the study of disequilibrium by defining a new sort of "non-Walrasian" fixed-price equilibrium, but this did not lead to a genuine disequilibrium analysis (Backhouse and Boianovsky, 2014; Dixon, 1990). In DSGE models, disequilibrium is only a temporary phenomenon after external shocks and does not arise from market interaction itself (Colander et al., 2008).

Our framework is neither limited to the assumption of (infinitely) fast price adaptation towards an equilibrium price, nor is the price fixed as in a "non-Walrasian" equilibrium. The stationary state reached is path dependent on the specific trading process before. In the example, we studied different price-adaptation processes by attributing different 'economic power' to the auctioneer, and allowed for trade during the slow price adjustments. While previous models thus often apply the extreme points of infinitely fast changing prices or fixed prices, general constrained dynamic models allow for a more realistic intermediate position.

5.4 Treat financial stocks and flows and their constraints consistently

After the recent financial crisis, it has been suggested to use macro-accounting frameworks, where the time evolution of financial stocks and flows is explicitly modeled (Bezemer, 2010). Similarly, Tobin (1982, p. 172) proposed to model financial assets as a "dynamic sequence" with "precision regarding time". Thereby, the "interdependence of asset markets enforced by balance-sheet relations" (Meyer, 1975, p. 65) has to be incorporated explicitly to avoid "pitfalls in financial model building" (Brainard and Tobin, 1968). In fact, these accounting identities and the necessity of stock-flow consistent modeling restrict the phase space reachable by the (model) economy (Caverzasi and Godin, 2015; Godley and Lavoie, 2012; Stützel, 1978), and can be integrated as constraints into GCD models straightforwardly.

5.5 Explicit formalization of economic power

Russell (2004, p. 108) argued in 1938 that economics is only "one element – a very important element, it is true – in a wider study, the science of power". In general equilibrium models, economic power is limited to the firms' "ability to establish a desirable price for its product in a particular market" (Spruill, 1983, p. 2). When markets are assumed to be perfectly competitive, even this form of 'market power' disappears, as no agent is able to influence the price. Bhaduri (2016, p. 32) argues that:

> "Those who set the 'rules of the game' by organizing the market and setting prices remain invisible like the force of gravity. So, like Voltaire's God, an auctioneer has to be invented to do the job of setting the market clearing prices. That invented God also prevents people from trading at non-market clearing prices. It is the story of an all-powerful auctioneer and powerless passive participants in the market."

In the exchange model, we made this power of the auctioneer explicit and illustrated that depending on whether the auctioneer is given no power, some power, or infinite power to influence prices different rest points are reached.

The concept of power as the ability to influence a certain variable may be applied to questions of political economy or direct interaction between people. Moreover, it may contribute to the discussion on 'closure', i. e. the task of making an under- or over-determined system of economic equations solvable. "... [P]rescribing closures boils down to stating which variables are endogenous or exogenous" (Taylor, 1991, p. 41), which constitutes the essential difference between economic schools of thought (Kaldor, 1955; Marglin, 1987; Sen, 1963). The choice of closure is a decision on the ability of different economic agents to influence certain variables and therefore reflects implicit assumptions on the power relations in the model. As an example, one does not have to specify whether saving S determines investment I (neoclassical closure) or inversely (Keynesian closure), but the definition $I - S = 0$ can simply be added as a constraint (Glötzl, 2015). Different closures can be reproduced by setting certain power parameters to 0 or ∞ (Glötzl, 2015). The framework allows to analyze different closures and 'mixed' power relations with joint influence of agents on direction and adaptation speed of specific variables.

5.6 Well-known general equilibrium solutions as fixed points of the dynamical system

GCD models are able to describe well-known equilibrium solutions as well as the out-of-equilibrium dynamics of the system. For our dynamic exchange model, we proved that for non-zero power factors the set of stationary states is identical to the standard static version of the model, but that the actual stationary state reached is path dependent.

From a general equilibrium perspective one may criticize that we replace the search for very general results about the existence and stability of a general equilibrium by rather arbitrary explicit dynamics. However, other assumptions are similarly arbitrary and quite restrictive: the macro-level restrictions on heterogeneity to allow for an representative agents approach (Rizvi, 1994), that price adaptation is much or infinitely faster than trade as in auctioneers models (Negishi, 1989), or that unmet demand and supply in one good never happen simultaneously as that market has already cleared sufficiently as in the Edgeworth or Hahn process (F. M. Fisher, 1983). Our approach may help to make these assumptions more explicit and to relax them.

5.7 Limitations

Several aspects remain to be shown in our framework, i. e. that production, heterogeneity, capital markets and plausible behavioral assumptions such as (limited) inter-temporal optimization, portfolio choice or strategic price setting can be consistently integrated. While stochastic shocks can easily be added to the time evolution, the integration of expectations and discounted utility functions without equilibrium assumptions would require (rather intricate) variational calculus. Current general equilibrium frameworks with multiple markets are tremendously complex in the amount of variables that are simultaneously 'in equilibrium'. Consequently, providing models able to describe genuine out-of-equilibrium dynamics for all these variables poses a significant challenge.

Some authors may be inclined to oppose our approach of searching for analogies between economics and mechanics *per se.* Grattan-Guinness (2010, pp. 571–2) argued that "we should protect economics from attractive but weak analogies with more established sciences", and "appealing to mechanics did and does not have much to offer to economics, so its revival is not to be encouraged". The difficulty in establishing economic models based on such analogies to physics was perfectly paraphrased by Murray Gell-Mann (Page, 1999, p. 36): "Imagine how hard physics would be if electrons could think."

We still use simple 'mechanic' behavioral rules, and the critique of economic Newtonianism and mechanism (Brodbeck, 2011; Mirowski, 1988, 1989) expressed in opposition to neoclassical economics may persist, in particular as we extend the approach of Pareto who was called the "most ruthless proponent of the physical metaphor" by Mirowski (1989, p. 221). On the other hand, as social interaction, power relations, and behavioral assumptions different from rationality can be integrated into our framework, we hope to obtain a more lenient sentence.

6 Conclusions

We presented a dynamic modeling approach in continuous time that suggests a route to extend existing analogies between mechanics and economics, as the latter has been hitherto limited to constrained optimization. Our framework is based on the concepts of *economic force*, *economic power*, and *economic constraint force*, and models dynamic interaction as well as genuine out-of-equilibrium evolution. This approach builds on constrained dynamics known from theoretical mechanics. Taking up the criticism to the general equilibrium approach, we developed the framework of general constrained dynamic models that can incorporate behavioral assumptions different from rationality and aggregation, slow price adaptation and out-of-equilibrium dynamics, consistently treat monetary assets, serve as a meta-modeling tool for agent-based models, and explicitly formalize economic power.

We illustrated numerous of these benefits to our approach with a dynamic version of a static textbook exchange model. This article provides only an intuition on the general constrained dynamics modeling framework, but alludes to the wide range of possibilities it entails. It may revive the fruitful exchange of ideas and concepts between physics and economics.

Acknowledgements

The three author names are displayed in alphabetical order. We thank Amit Bhaduri, Armon Rezai, Galo Nuno, Hans-Michael Trautwein, Heinz D. Kurz, Jakob Kapeller, and Michael-Gregor Miess for their helpful comments. All errors and omissions are our own. OR acknowledges financial support from Ev. Studienwerk Villigst. Conflicts of interest: none.

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