

Decision and Choice: Random Utility Models of Choice and Response Time

Hans Colonius, Carl von Ossietzky Universität Oldenburg, Oldenburg, Germany

Anthony AJ Marley, University of Victoria, Victoria, BC, Canada; and Institute for Choice, University of South Australia, SA, Australia

© 2015 Elsevier Ltd. All rights reserved.

Abstract

Horse race random utility models for choice and response time are studied, and conditions on the observable choice probabilities and decision times necessary and sufficient for a random utility representation are investigated. We state conditions for stochastic independence of option chosen from time of choice. Context-free accumulator models, like the linear ballistic accumulator model, are presented as horse race models with additional structure. Models with multiplicative and with additive drift rate variability are discussed.

Introduction

The task of choosing a single 'best' option from some available, potentially infinite, set of options A has received considerable study in psychology and economics. *Random utility models* of choice account for the stochastic variability underlying these choices (i.e., the same choice is not necessarily made on repeated presentations of the same set of options) by assuming that there exists a random variable, $U(x)$ for each option, x and a joint probability measure for these random variables such that the probability of choosing a particular option, x from the set of available options is equal to the probability that $U(x)$ takes on a value greater than the values of all other random variables (see Decision and Choice: Luce's Choice Axiom). The basic choice paradigm is extended here by considering, in addition to the option chosen, the point in time that a choice is made from the set of available options. The random utility model can be tailored to cover this situation by replacing the utility variable, $U(x)$ by $T(x) = \phi[U(x)]$ (ϕ , some monotonically decreasing transformation of the values of U into the nonnegative real numbers), where $T(x)$ can be interpreted as the decision time for choosing option x , and by replacing the maximum utility rule by a minimum decision time rule: the option chosen is the one that happens to be associated with the minimum choice (or decision) time with respect to all options in the available set. This model will be referred to as a *horse race random utility model*. Over the past 15 years, psychologists and economists have extended the random utility framework to tasks where a person might be asked to select the 'worst' option, or the 'best' and 'worst' option, from some available choice set. For reasons of space, we do not present parallel results for these cases, but see the results in Hawkins et al. (2014) and the summaries in Marley and Flynn (2015) and Marley and Regenwetter, (in press).

A number of questions arise in the study of horse race random utility models for choice and response time, some of which have received partial answers: (1) What conditions on the observable choice probabilities and decision times are necessary and sufficient for a random utility representation? (2) What are the consequences of assuming stochastic independence between the time of choice and the identity of the option chosen, and which assumptions on the joint distribution function imply this independence? (3) What are possible generalizations to other choice paradigms? The presentation

here is partly based on Marley (1989) and Marley and Colonius (1992); those papers present and cite related results that are not explicitly referred to in the following.

Horse Race Random Utility Models

Let $A = \{x, y, \dots\}$ be a *finite* set of potential choice options (for an extension of the theory to infinite choice sets see Resnick and Roy, 1992); the subset X of A containing at least two elements is the currently available choice set. T_X is a random variable denoting the time at which a choice is made; and B_X is a random variable denoting the element chosen from X . For $t \geq 0$,

$$B_X(x; t) = \Pr[B_X = x \cap T_X > t] \quad [1]$$

is the probability that option x is chosen as the best from X after time t , and $\{B_X(x; t) : x \in X \subseteq A\}$, or (A, P) for short, is called a *joint structure of choice probabilities and response times*. (A, P) is called *complete* if it is defined for all subsets of X of A with $|X| \geq 2$.

A complete joint structure of choice probabilities and response times, (A, P) is said to satisfy a *horse race random utility model* if there exists a probability measure, $\Pr_X[\]$ for the collection of nonnegative random variables, $\{T(x) : x \in X\}$ such that, for all $X \subseteq A$ with $|X| \geq 2$,

$$B_X(x; t) = \Pr_X[t < T(x) < T(y) : y \in X \setminus \{x\}] \quad [2]$$

For simplicity, all distributions are assumed to be absolutely continuous, i.e., they must possess a density function; however, as observed in Resnick and Roy (1992), absolute continuity is not necessary for the results cited here. The representation eqn [2] is of the form given in eqn [1] with $T_X = \min\{T(y) : y \in X\}$. Also note that with

$$\begin{aligned} B_X(x) &= \lim_{t \downarrow 0} \Pr_X[t < T(x) < T(y) : y \in X \setminus \{x\}] \\ &= \Pr_X[T(x) < T(y) : y \in X \setminus \{x\}] \end{aligned}$$

the collection of choice probabilities, $\{B_X(x) : x \in X \subseteq A\}$ constitutes a *system of choice probabilities* with

$$B_X(x) = \Pr_X[T(x) = \min\{T(y) : y \in X\}]$$

satisfying a *random utility representation* with min replacing the usual max.

In this article the problem of finding conditions on the – at least theoretically – observable choice probabilities and

response time distributions, which are necessary and sufficient for the existence of a particular representation in terms of a collection of underlying random variables, is referred to as a *characterization problem*.

General characterization problem. Given a (complete) joint structure of choice probabilities and response times what conditions on the (survival) functions, $B_X(x;t)$ are necessary and sufficient for the existence of a (possibly dependent) horse race random utility representation for that joint structure?

A complete answer to this problem is still open. However, a number of results in special cases have been found.

Representation by Independent Random Variables

The following is a partial answer to the general characterization problem:

Theorem 1 (Marley and Colonius, 1992).

- a. For a collection of choice probabilities and response times, $\{B_X(x;t) : x \in X\}$, on a fixed set X with $|X| \geq 2$, there exist independent random variables, $T(x)$, $x \in X$, with unique distributions, such that eqn [2] holds, provided $B_X(x;t)$ is absolutely continuous and positive for all $t \geq 0$.
- b. A complete joint structure of choice probabilities and response times (A, P) can be represented by an independent horse race random utility model, with unique distributions, if the conditions of (a) hold and

$$\left[\sum_{y \in X} B_X(y;t) \right]^{-1} (d/dt)B_X(x;t) \quad [3]$$

is independent of $X \subseteq A$ for all $t \geq 0$.

Thus, given the above regularity conditions on the response time distributions, part (a) shows that any set of choice probabilities and response times on a fixed finite set, X can be represented by independent random variables such that the horse race eqn [2] holds; and part (b) gives sufficient conditions for a common representation across all subsets of A . Townsend (1976) develops a related result in the context of parallel-serial processing analysis. Thus, the general characterization problem is still unsolved leaving open the possibility that for some joint structures no independent representation over all subsets X exists.

Some of the distributions, but not all, in the above solutions may be *improper*, i.e., they may have positive measure at infinity implying an infinite decision time for that alternative, even if the distribution of response time, T_X is finite almost surely. Jones and Dzhafarov (2014) present related interpretations and further falsifiability results for horse race and other models of choice and response time.

The above framework has natural extensions to subset choice (Marley and Colonius, 1992) and to the worst and best-worst choice (Marley and Regenwetter, in press). Also, in the theory of competing risks (e.g., David and Moeschburger, 1978) the 'event' associated with each $x \in X$ is reinterpreted as a *cause* (e.g., of failure or death) and thus the selection of $x \in X$ then corresponds to the cause associated with x being the first to occur when X contains the possible causes. This leads to *cause-specific* (or *crude hazard rate*) interpretations of the results and related *nonidentifiability* issues (Marley and Colonius, 1992).

Independence of Option Chosen from Time of Choice

A classic model for best choice is *Luce's choice model* (see Decision and Choice: Luce's Choice Axiom), also called the *multinomial logit (MNL) model* (Marley and Regenwetter, in press). That model holds for a system of choice probabilities, $\{B_X(x) : x \in X \subseteq A\}$ provided there is a ratio scale b on A such that for $B_X(x) \neq 0, 1$

$$B_X(x) = \frac{b(x)}{\sum_{y \in X} b(y)} \quad [4]$$

For simplicity in the following, we assume that all choice probabilities are nonzero. The result can be generalized when this is not the case by adding a connectivity and a transitivity condition (Luce, 1959: Theorem 4, p. 25).

Theorem 2 (Marley and Colonius, 1992)

Consider a (complete) independent horse race random utility model, (A, P) where for each $x \in X \subseteq A$ with $|X| \geq 2$, $B_X(x;t)$ is positive and absolutely continuous for all $t > 0$. If T_X is stochastically independent of B_X , then the choice probabilities satisfy Luce's choice model.

Given the known limited empirical validity of Luce's choice model in empirical preference situations, this result implies that one must study *dependent* horse race random utility models and/or drop the assumption of independence between T_X and B_X . We now present results for such generalizations.

Generalized Stable Survival Functions

A large class of (possibly) dependent horse race random utility models based on extreme value distributions can be generated from the concept of a *generalized stable survival function*. For simplicity, in this section, assume $A = \{x_1, \dots, x_n\}$. We call

$$P_A(t_1, \dots, t_n) = Pr[T(x_1) > t_1, \dots, T(x_n) > t_n]$$

a (*multivariate*) survival function. P_A is called *generalized stable* if there is a strictly monotone decreasing function, η and a constant, $\mu > 0$ such that for all $\alpha > 0$, $t_i \geq 0$, $i = 1, \dots, n$,

$$(\eta \circ P_A)(\alpha t_1, \dots, \alpha t_n) = \alpha^\mu (\eta \circ P_A)(t_1, \dots, t_n) \quad [5]$$

where \circ denotes the concatenation of functions. Letting $G_A = (\eta \circ P_A)$, this amounts to

$$G_A(\alpha t_1, \dots, \alpha t_n) = \alpha^\mu G_A(t_1, \dots, t_n)$$

i.e., G_A is homogeneous of degree μ . Furthermore, a survival function P_A is called a *strictly monotone transform of a generalized stable survival function*, Q_A if there is a strictly monotone increasing function γ (of the random variables associated with Q_A) with $\gamma(0) = 0$, $\gamma(\infty) = \infty$ such that for $t_i \geq 0$, $i = 1, \dots, n$,

$$P_A(t_1, \dots, t_n) = Q_A[\gamma(t_1), \dots, \gamma(t_n)]$$

Theorem 3 (Marley, 1989)

Any horse race random utility model that is generated by a strictly monotone transform of a generalized stable survival function is such that T_X is stochastically independent of B_X for any $X \subseteq A$.

Resnick and Roy (1992: Section 5) construct an example, which shows that the converse of this result does not hold in general, i.e., independence between T_X and B_X does not lead back to the class of generalized stable survival functions. However, Robertson and Strauss (1981) show that the converse is true if the survival function, P_A (or some strictly monotone transformation of it) belongs to the generalized Thurstone class: for $t_i \geq 0, i = 1, \dots, n$,

$$P_A(t_1, \dots, t_n) = P(u(x_1)t_1, \dots, u(x_n)t_n)$$

where $u(x_i) \geq 0$ and P is a survival function that is independent of A .

Marley (1989) uses generalized stable survival functions to extend a large class of frequently applied random utility models for choice to models of choice and response time; these models of choice include McFadden's *generalized extreme value* class, Tversky's *elimination by aspects* model, and Strauss's *general feature* model. However, by the above results, all these models predict the independence of the choice made and the time to make it, and so, in particular, that the response time distribution is the same for all choices. Given that the latter condition is not satisfied by various data (Hawkins et al., 2014), we now turn to related models that do not have this property.

The results above were stated in terms of the *survival* form

$$B_X(x; t) = \Pr(B_X = x \text{ and } T_X > t)$$

If we now let $B_X(x, t)$ denote the probability that option x is chosen *at or before* time t (note the change of the semicolon ; to a comma ,), then those results can be written equally well in the *cumulative* form

$$B_X(x, t) = \Pr(B_X = x \text{ and } T_X \leq t) \quad [6]$$

Since most response time models are written in terms of such cumulative distributions, we will use the latter form in the remaining sections.

Context-Free Linear Accumulator Models

Context-free accumulator models are essentially horse race random utility models with additional structure imposed on the properties of the random variables; later we summarize context-dependent accumulator models. Here, we discuss the *linear ballistic accumulator* (LBA) model (Heathcote and Love, 2012). Like other multiple accumulator models, the LBA is based on the idea that the decision maker accumulates evidence in favor of each choice, and makes a decision as soon as the evidence for any choice reaches a threshold amount. The time to accumulate evidence to threshold is the predicted decision time, and the response time is the decision time plus a fixed offset (t_0), the latter accounting for processes such as response production. The LBA shares this general evidence accumulation framework with many models (Busemeyer and Rieskamp, 2013; see Diffusion and Random Walk Processes) but has a practical advantage – namely, the computational tractability afforded by having an easily computed expression for the joint likelihood of a given response time and response choice among any number of options.

In the following sections, we consider two classes of LBA – the first class relates in a natural way to the models already presented, the second class agrees with the assumptions in the

original development of the LBA. We first summarize context-independent models, and then summarize context-dependent extensions; we ignore the fixed offset, (t_0).

Multiplicative Drift Rate Variability

Let Δ be a random variable on the positive real numbers; this will be the distribution of the (multiplicative) drift rate variability, where *drift rate* is the rate of accumulation of evidence. Let, for $r > 0$,

$$\Pr(\Delta \leq r) = G(r) \quad [7]$$

where G is a cumulative distribution function (CDF). Let b denote the threshold, x a typical choice option with drift rate $d(x)$, and Σ a random variable on $[0, A]$ with $A \leq b$; this will be the distribution of the start point of the evidence accumulation. Let, for $h \in [0, A]$,

$$\Pr(\Sigma \leq h) = H(r)$$

where H is a CDF.

For option x , the start point has distribution, Σ and the drift rate has distribution, $\Delta d(x)$. We assume that the samples of Σ and Δ for different values of x are independent, which we denote by Σ_x and Δ_x ; thus, the subscripts do not imply dependence on x beyond the assumption of independent samples for different values of x . Then, with b_x denoting the random variable for the time taken for the LBA with (mean) drift rate, $d(x)$ to reach threshold, we have

$$\Pr(b_x \leq t) = \Pr\left(\frac{b - \Sigma_x}{\Delta_x d(x)} \leq t\right) \quad [8]$$

The major LBA applications to date have assumed that the start point variability is given by a random variable, p with uniform distribution on $[b-A, b]$, in which case eqn [8] can be rewritten as

$$\Pr(b_x \leq t) = \Pr\left(\frac{p_x}{\Delta_x d(x)} \leq t\right) \quad [9]$$

Then the choice and response time representation, $B_X(x, t)$ of eqn [6] holds with $T_X = \min\{b_y; y \in X\}$.

The above representation is a perfectly plausible multiplicative LBA model for *any* cumulative distribution G on the nonnegative reals and H on $[0, A]$ with $A \leq b$. However, a major argument advanced for LBA models is their computational tractability in terms of *probability density functions* (PDFs) and CDFs. In particular, it is usually assumed that the accumulators are independent, and therefore the main need is for the PDF and CDF of eqn [9] to be tractable. We summarize the components of the CDF corresponding to that expression, and then summarize what is known about such forms.

With p , the uniform distribution on $[b-A, b]$, and the CDF of G given by eqn [7], standard calculations show that $\Pr(b_x \leq t)$ of eqn [9] can be written in terms of: the constants b and A ; the mean of the distribution G when truncated to the interval, $[b/t d(x), b-A/t d(x)]$; and the values of $G[b-A/t d(x)]$ and $G[b/t d(x)]$ (Marley and Regenwetter, in press). Closed forms are known for the first quantity for various CDFs on the nonnegative reals, including for Gamma, inverted Beta, Fréchet, and Levy distributions (Nadarajah, 2009). However, one of the major motivations for the LBA framework was to achieve easily computable forms for the joint distribution of choice

probabilities and choice times – i.e., for eqn [6]. In addition to independence, this requires the PDFs corresponding to the CDFs, above, to be easily computable. Ongoing research is exploring the extent to which this is the case for various distributional assumptions. For example, [Heathcote and Love \(2012\)](#) develop and test the tractable *lognormal race*, which has the form: for each option x ,

$$\Pr(b_x \leq t) = \Pr\left(\frac{S_x}{D_x} \leq t\right)$$

where S_x is the distribution to the boundary and D_x is the drift rate distribution, with each being lognormally distributed.

Relation of Multiplicative LBA Models with No Start Point Variability and Fréchet Drift Rate Variability to Luce (MNL) Models

We now present a multiplicative LBA with no start point variability and Fréchet drift scale variability that leads to the Luce (MNL) model for the best choice; parallel results hold for the worst and best–worst choice ([Marley and Regenwetter, in press](#)). Consider the special case of eqn [9] where there is no start point variability, i.e., $A = 0$, and so p has a constant value, b ; in this case, without loss of generality, we can set $b = 1$. Then eqn [9] reduces to

$$\Pr(B_x \leq t) = \Pr\left(\frac{1}{\Delta d(x)} \leq t\right)$$

We also assume that Δ has a Fréchet distribution, i.e., there are constants $\alpha, \beta > 0$ such that, for $r \geq 0$,

$$\Pr(\Delta \leq r) = G(r) = e^{-(\alpha r)^\beta} \quad [10]$$

Then eqn [6] is (remember that the subscript on Δ_z indicates independent samples, not other dependence on z):

$$B_X(x, t) = \Pr\left(\frac{1}{\Delta_x d(x)} = \min_{z \in X} \frac{1}{\Delta_z d(z)} \leq t\right) \quad [11]$$

For $t \geq 0$, let $F(t) = 1 - \exp\left(-\left(\sum_{z \in X} d(z)^\beta\right)(\alpha t)^\beta\right)$. Then routine calculations (paralleling those in [Marley, 1989](#)) give: for $X \subseteq A$ and $x, y \in X$,

$$B_X(x; t) = \frac{d(x)^\beta}{\sum_{z \in X} d(z)^\beta} F(t) \quad [12]$$

The choice probabilities, $B_X(x)$ are given by this formula in the limit as $t \rightarrow \infty$, i.e., as $F(t) \rightarrow 1$, which shows that they satisfy Luce’s choice model, eqn [4], with $b(z) = d(z)^\beta$. Now, for $z \in A$, let $u(z) = \ln d(z)$, $\varepsilon_z = \ln \Delta_z$. Then the form for the choice probabilities can equally well be written as: for all $x, y \in X \in D(A)$,

$$B_X(x) = \Pr\left\{u(x) + \varepsilon_y = \max_{z \in X} [u(z) + \varepsilon_z]\right\} \quad [13]$$

However, given that Δ_z and $\Delta_{p,q}$ are generated by Fréchet drift rate variability, i.e., eqn [10], we have that $\varepsilon_z = \ln \Delta_z$ and $\varepsilon_{p,q} = \ln \Delta_{p,q}$ satisfy extreme value distributions (with, in this case: for $-\infty < t < \infty$ $\Pr(\varepsilon_z \leq t) = \exp\left(-(\alpha e^t)^{-\beta}\right)$ and $\Pr(\varepsilon_{p,q} \leq t) = \exp\left(-(\alpha e^t)^{-\beta}\right)$). Under these conditions, eqn [13] corresponds to the usual representation of Luce’s choice model as a random utility model ([Marley and Louviere, 2008](#), used

the case $\alpha = \beta = 1$). Note that β is not identifiable from the choice probabilities, but it is, in general, identifiable when response times are also available (though the form predicted for the response time distribution is not suitable for data).

As already shown by eqn [12], this model has the property that the option chosen is independent of the time of choice; that property would not hold for (most) distributions other than the Fréchet for drift scale variability, nor does it hold when the above model is extended to include start point variability.

Additive Drift Rate Variability

In contrast to the multiplicative drift rate model presented above, the *LBA model* assumes *additive* drift rate variability generated by independent normal random variates, truncated at zero – i.e., constrained to be nonnegative ([Heathcote and Love, 2012](#)). This model is a horse race random utility model, but does not have the undesirable property that the choices made and the time to make them are independent. Nonetheless, it has been shown to make extremely similar predictions for the best, worst, and best–worst choice probabilities to those made by the multiplicative LBA model with no start point variability and Fréchet drift scale variability ([Hawkins et al., 2014](#)). The derivations for this model exactly parallel those for the multiplicative LBA model, above, with $\Delta d(x)$ replaced by $\text{trunc}[D_z + d(z)]$, with a major advantage of the additive model being that all formulas for CDFs and PDFs are computationally very tractable when the start point distribution is uniform. [Marley and Regenwetter \(in press\)](#) briefly summarize recent applications of this model to some standard data on the best, worst, and best–worst choice and its extension to handle context effects in the best choice and response times.

Context-Dependent Models of Response Time and Choice

As far as we know, there is no formal definition of what it means for a set of (theoretical or empirical) choice probabilities to demonstrate a *context effect*. The general usage of that term appears to be that the choice probabilities are not consistent with either a *random utility* or a *constant utility* representation as defined in Chapter 10 of [Suppes et al. \(1989\)](#). [Busemeyer and Rieskamp \(2013\)](#) and [Rieskamp et al. \(2006\)](#) summarize standard context effects and the extant models that are, or are not, compatible with those effects holding. They do not discuss the recent *multiattribute linear ballistic accumulator (MLBA) model* ([Trueblood et al., 2014](#)) or the *2N-ary tree model* ([Wollschläger and Diederich, 2012](#)), to which we now turn.

[Trueblood et al. \(2014\)](#) show that their *context-dependent* MLBA model can produce various standard context effects in choice and various properties of the context effects when participants are under time pressure. These results are obtained by adding a context-dependent ‘front end’ to a context-independent LBA model. This suggests taking a fresh look at various choice-only models that can handle some or all such context effects in choice, and extending them to response time by, say, adding a context-independent LBA model that is driven by the ‘front end’ parameters of the choice model (see [Marley and Regenwetter, in press](#)).

The *2N-ary choice tree model* for N-alternative preferential choice ([Wollschläger and Diederich, 2012](#)) builds on earlier stochastic dynamic models (*accrual-halting models*, [Townsend](#)

et al., 2012) and thus includes details of the temporal accumulation of information. The structures, though not the details, of the $2N$ -ary choice tree model and of the context-dependent MLBA model are rather similar – the major differences are between the ballistic accumulators of the MLBA model and the transition probabilities of the $2N$ -ary choice tree model, and the fact that the MLBA model has no decay, whereas the $2N$ -ary choice tree model has decay. Each model well fits the data presented by its authors.

To date, the $2N$ -ary choice tree model has been applied to the best choice, though it includes a mechanism for rejecting options. Thus, it is of interest to extend the $2N$ -ary choice tree model to the best–worst choice and compare its fits to the data from such choice with those of the *parallel best–worst LBA model* (Hawkins et al., 2014).

Discussion and Conclusion

In contrast to the context-dependent MLBA model, the $2N$ -ary choice tree model and related stochastic dynamic models specify details of the temporal accumulation of information, rather than having a front end that incorporates the overall effect of such accumulated information. A significant part of the reason such stochastic dynamic models have not been used in applied areas is the “relatively high difficulty of estimating some sequential sampling models of decision making” (Berkowitsch et al., 2013). This difficulty is being alleviated, for example, by Berkowitsch et al.’s (2013) derivation of a tractable closed form for the choice probabilities given by an asymptotic form of decision field theory and by the various LBA models for (best, worst, best–worst) choice and response times.

The context-dependent models of choice and response time presented in this article involve numerous complex processes. Further empirical study along the lines of Teoderescu and Usher (2013), with follow-up theoretical work, is needed to clarify the conditions under which some or all these, and possibly other, processes are needed.

Nonetheless, the introduction of the notion of response (or decision) time into the random utility approach to modeling choice behavior has been very fruitful. It has brought about new insights into existing stochastic choice models and their characterizations, and it has generated new questions, many of which have only found partial answers up to now.

See also: Decision and Choice: Luce’s Choice Axiom; Diffusion and Random Walk Processes.

Bibliography

- Berkowitsch, N.A.J., Scheibehenne, B., Rieskamp, J., 2013. Testing multialternative field theory rigorously against random utility models, Manuscript.
- Busemeyer, J.R., Rieskamp, J., 2013. Psychological research and theories on preferential choice. In: Hess, S., Daly, A. (Eds.), *Handbook of Choice Modelling*. Edward Elgar Publishing.
- David, H.A., Moeschburger, M.L., 1978. *The Theory of Competing Risks*. Griffin, London.
- Hawkins, G., Marley, A.A.J., Heathcote, A., Flynn, T., Louviere, J., Brown, S.D., 2014. Integrating cognitive process and descriptive models of attitudes and preferences. *Cognitive Science* 38, 701–735.
- Heathcote, A., Love, J., 2012. Linear deterministic accumulator models of simple choice. *Frontiers in Psychology* 3, 1–19.
- Jones, M., Dzhafarov, E.N., 2014. Unfalsifiability and mutual translatability of major modeling schemes for choice reaction time. *Psychological Review* 121, 1–32.
- Luce, R.D., 1959. *Individual Choice Behavior*. Wiley, New York.
- Marley, A.A.J., 1989. A random utility family that includes many of the “classical” models and has closed form choice probabilities and reaction times. *British Journal of Mathematical and Statistical Psychology* 42, 13–36.
- Marley, A.A.J., Flynn, T.N., 2015. Best worst scaling: theory and practice. In: Wright, James D. (Editor-in-Chief), *International Encyclopedia of the Social and Behavioral Sciences*, second ed. Elsevier, Oxford.
- Marley, A.A.J., Louviere, J.J., 2008. Some probabilistic models of best, worst, and best–worst choice. *Journal of Mathematical Psychology* 44, 464–480.
- Marley, A.A.J., Regenwetter, M. Choice, preference, and utility: probabilistic and deterministic representations. In: Batchelder, W., Colonius, H., Dzhafarov, E., Myung, J. (Eds.), *New Handbook of Mathematical Psychology*. Cambridge University Press, Cambridge, in press.
- Marley, A.A.J., Colonius, H., 1992. The horse race random utility model for choice probabilities and reaction times, and its competing risks interpretation. *Journal of Mathematical Psychology* 36, 1–20.
- Nadarajah, S., 2009. Some truncated distributions. *Acta Applied Mathematics* 106, 105–123.
- Resnick, S.I., Roy, R., 1992. On min-stable horse races with infinitely many horses. *Mathematical Social Sciences* 23, 119–145.
- Rieskamp, J., Busemeyer, J., Mellers, B., 2006. Extending the bounds of rationality: evidence and theories of preferential choice. *Journal of Economic Literature* 44, 631–661.
- Robertson, C.A., Strauss, D.J., 1981. A characterization theorem for random utility variables. *Journal of Mathematical Psychology* 23, 184–189.
- Suppes, P., Krantz, D.H., Luce, R.D., Tversky, A., 1989. *Foundations of Measurement*, vol. II. Academic Press, San Diego, CA.
- Teoderescu, A., Usher, M., 2013. Disentangling decision models – from independence to competition. *Psychological Review* 120, 1–38.
- Townsend, J.T., 1976. Serial and within-stage independent parallel model equivalence on the minimum completion time. *Journal of Mathematical Psychology* 14, 219–238.
- Townsend, J., Hout, J., Silbert, N., 2012. General recognition theory extended to include response times: predictions for a class of parallel systems. *Journal of Mathematical Psychology* 56, 476–494.
- Trueblood, J.S., Brown, S.D., Heathcote, A., 2014. The multiattribute linear ballistic accumulator model of context effects in multialternative choice. *Psychological Review* 121, 179–205.
- Wolfschläger, L., Diederich, A., 2012. The $2n$ -ary choice tree model for n -alternative preferential choice. *Frontiers in Psychology* 3 (189).