Reflected Brownian Motion and Local Time

Diploma Thesis

by

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Contents

1. Introduction

A classical problem in financial mathematics is how to invest and how to consume optimally when having a bank account paying a fixed interest rate r and a risky asset, modeled as geometric Brownian motion. For some utility functions, it has been shown that in the absence of transaction cost it is optimal to keep a constant fraction π^* of total wealth in the risky asset and to consume at a rate proportional to total wealth. A complete description and the proof can be found in [DAVIS and NORMAN 1990, Theorem 2.1]. The line of portfolios corresponding to the fraction π^* is called *Merton line*. When considering proportional transaction costs, applying Merton's strategy would result in immediate bankruptcy and it would become optimal to keep the fraction of wealth in stocks and total wealth within an interval. It can be shown that this *no-transaction region is a wedge containing the Merton line*, see [DAVIS and NORMAN 1990, Theorem 3.1]. This leads to the optimally controlled process being a reflected diffusion inside the wedge and the buying and selling policies being the local times at the lower and upper boundaries, respectively, for a proof see [DAVIS and NORMAN 1990, Theorem 4.1].

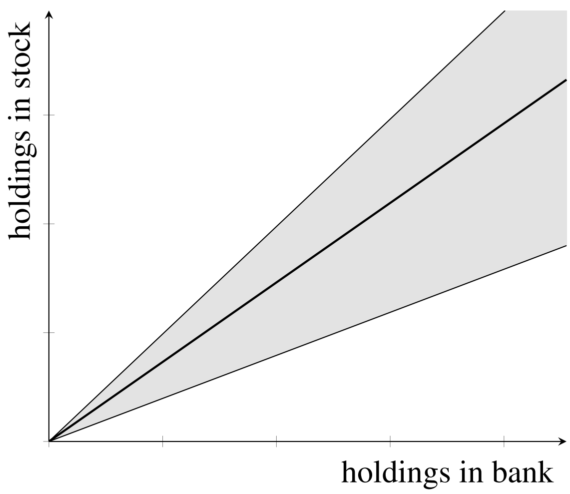

Figure 1.1.: Space of bank and stock holdings with Merton line —— and no-transaction region (shaded in gray).

This work is devoted to the existence and uniqueness of these *reflected diffusions*. We start with the one-dimensional setting that has originally been studied by A. Skorokhod in Chapter 3, correspondingly this kind of problem is called *Skorokhod problem*. In Chapter 4 we consider conditions ensuring the existence and uniqueness in a multi-dimensional setting when the reflection is along a normal vector on the surface. This can be generalized and we give some results concerning this so called oblique reflection in Chapter 5 and cover the example of the positive orthant in Section 5.2 in detail. We finish in Chapter 6 with the generalization to time-dependent boundaries, for this we restrict ourself again to the one-dimensional case.

2. Theoretical Basics

The following chapter is meant as a brief introduction into Brownian motion and follows roughly [KARATZAS and SHREVE 2000], which is a good source to get a more thorough insight into the topic. Our survey concentrates mainly on the results we are going to apply later on.
In the entire chapter we will consider a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ equipped with a filtration $\mathscr{F} = (\mathscr{F}_t)_{t \geq 0}$.

2.1. Brownian motion

Definition 2.1 (Brownian motion)**.** *A* (standard, one-dimensional) Brownian motion *is an $\mathscr{F}$-adapted process $W = (W_t)_{t \geq 0}$ satisfying the following properties:*

1. $W_0 = 0$ $\mathbb{P}$*-a.s.*
2. *W is $\mathbb{P}$-a.s. continuous*
3. *For $\tilde{t} < t \in \mathbb{R}^+$ the increment $W_t - W_{\tilde{t}}$ is independent of $\mathscr{F}_{\tilde{t}}$ and normally distributed with mean zero and variance $t - \tilde{t}$.*

Before we start to list some properties of Brownian motion, we just quote the following result concerning the existence of Brownian motion.

Theorem 2.2 (Existence of Brownian motion)**.** *There is a probability space $(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{\mathbb{P}})$ equipped with a filtration $\mathscr{F}$ and an $\mathscr{F}$-adapted stochastic process W on it, such that W is a standard one-dimensional Brownian motion.*

Proof. The proof can be found in [KARATZAS and SHREVE 2000, Section 2.2]. □

Throughout Chapter 3 we will consider $\max_{s \leq t} W_s$ and we need the joint distribution of W_t and $\max_{s \leq t} W_s$, to obtain it we apply the following result, closely related to the strong Markov property of Brownian motion.

Lemma 2.3. *Let $\tilde{\tau}$ and τ be two $\mathscr{F}$-stopping times, such that $\tilde{\tau} \leq \tau$, and $a \in \mathbb{R}$. Then, we have for $\mathbb{P}$-almost all $\omega \in \{\tau < \infty\}$*

$$\mathbb{P}(W_\tau \leq a | \mathscr{F}_{\tilde{\tau}+})(\omega) = \mathbb{P}^{W_{\tilde{\tau}(\omega)}(\omega)}(W_{\tau - \tilde{\tau}} \leq a), \tag{2.1}$$

where $\mathbb{P}^x$ is the probability measure associated to the Brownian motion starting in $W_0 = x$.

Proof. The proof can be found in [KARATZAS and SHREVE 2000, Corollary 2.6.18]. □

To calculate the joint distribution, we let $t > 0$ and $a \leq b$, $b \geq 0$. The first step is to note that the symmetry of Brownian motion implies

$$\mathbb{P}^b(W_{t-\tilde{t}} \leq a) = \mathbb{P}^b(W_{t-\tilde{t}} \geq 2b-a). \tag{2.2}$$

We want to additionally have the condition $\max_{s\leq t} W_s \leq b$, to see that (2.2) remains valid we have to apply Lemma 2.3 to the *passage time* $T_b = \inf\{t \geq 0; W_t = b\}$:

$$\begin{aligned}\mathbb{P}(W_t \leq a|\mathscr{F}_{T_b+}) &\overset{(2.1)}{=} \mathbb{P}^{W_{\tilde{\tau}(\omega)}(\omega)}(W_{\tau-\tilde{\tau}} \leq a)\\ &\overset{(2.2)}{=} \mathbb{P}^{W_{\tilde{\tau}(\omega)}(\omega)}(W_{\tau-\tilde{\tau}} \geq 2b-a)\\ &\overset{(2.1)}{=} \mathbb{P}(W_t \geq 2b-a|\mathscr{F}_{T_b+}) \text{ } \mathbb{P}\text{-a.s. on } \{T_b \leq t\}.\end{aligned}$$

Now, we can integrate both sides over $\{T_b \leq t\} = \{\max_{s\leq t} W_s \geq b\}$ to obtain

$$\begin{aligned}\mathbb{P}(W_t \leq a, \max_{s\leq t} W_s \geq b) &= \mathbb{P}(W_t \geq 2b-a, \max_{s\leq t} W_s \geq b)\\ &= \mathbb{P}(W_t \geq 2b-a)\\ &= \int_{2b-a}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-z^2/2t\right) dz.\end{aligned}$$

Now we differentiate and obtain the joint density of W_t and $\max_{s\leq t} W_s$

$$\mathbb{P}(W_t \in dy, \max_{s\leq t} W_s \in dx) = \sqrt{\frac{2}{\pi t^3}}(2x-y)\exp\left(-(2x-y)^2/2t\right) dy dx. \tag{2.3}$$

2.2. Local Time

Next, we will define *local time*. There are several natural ways to define it. We will follow the approach chosen in [KARATZAS and SHREVE 2000] and start by defining the *occupation time*

$$\Gamma_t(B) := \lambda(s \leq t; W_s \in A) = \int_0^t 1_A(W_s) ds \text{ for } B \in \mathscr{B}(\mathbb{R}).$$

Definition 2.4 (Local Time)**.** *Let $\lambda = (\lambda_t(x)(\omega); (t,x) \in [0,\infty) \times \mathbb{R}, \omega \in \Omega)$ be a random variable with values in $[0,\infty)$, such that the random variable $\lambda_t(x)$ is continuous and $\mathscr{F}_t$-measurable for every pair $(t,x) \in [0,\infty) \times \mathbb{R}$. If λ_t is the density of the occupation times Γ_t almost surely, i.e.*

$$\Gamma_t(A)(\omega) = \int_A \lambda_t(x)(\omega) dx \text{ for all } t \geq 0, \text{ and } B \in \mathscr{B}(\mathbb{R}), \tag{2.4}$$

for $\mathbb{P}$-almost all $\omega \in \Omega$, then $\lambda_t(x)$ is called the local time *of W at $x \in \mathbb{R}$.*

From (2.4) we can derive the explicit representation of local time

$$\lambda_t(x) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t 1_{(x-\varepsilon,x+\varepsilon)}(W_s) ds \tag{2.5}$$

by noting that for $\varepsilon > 0$

$$\begin{aligned}&1_{(x-\varepsilon,x+\varepsilon)}(W_s(\omega)) = 1\\ \iff &1_{(W_s(\omega)-\varepsilon, W_s(\omega)+\varepsilon)}(x) = 1,\end{aligned}$$

implying that for any set $A \in \mathscr{B}(\mathbb{R})$

$$\int_A 1_{(x-\varepsilon,x+\varepsilon)\cap A}(W_s(\omega))dx = \begin{cases} 2\varepsilon & \text{for} \quad d(W_s(\omega),A^C) \geq \varepsilon \\ 0 & \text{for} \quad d(W_s(\omega),A) \geq \varepsilon, \end{cases}$$

thus, if $W_s(\omega)$ is not on the boundary of A

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_A 1_{(x-\varepsilon,x+\varepsilon)\cap A}(W_s(\omega))dx = 1_A(W_s(\omega))$$

This identity holds almost surely, as $\mathbb{P}(W_s(\omega) \in \partial A) = 0$. We can put this together to obtain

$$\begin{aligned} &\int_A \left(\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t 1_{(x-\varepsilon,x+\varepsilon)\cap A}(W_s)ds \right) dx \\ &= \int_0^t \left(\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_A 1_{(x-\varepsilon,x+\varepsilon)\cap A}(W_s)dx \right) ds \\ &= \int_0^t 1_A(W_s)ds \end{aligned}$$

This proves that $x \mapsto \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t 1_{(x-\varepsilon,x+\varepsilon)}(W_s)ds$ is the density of the occupation time and thus the local time.

3. One-dimensional diffusions with reflecting boundaries

Following the original paper of A. V. Skorokhod [Skorokhod 1961] we shall start by considering a one-dimensional diffusion $X = (X_t)_{t\geq 0}$, which in absence of a boundary can be described as the solution of the stochastic differential equation (SDE) depending on the diffusion coefficient σ and the drift coefficient μ

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \text{ for } t \geq 0, \tag{3.1}$$

where $W = (W_t)_{t\geq 0}$ denotes the standard Brownian motion and both coefficients are known functions of time t and the corresponding value of X. The existence and uniqueness of a strong solution X can be proven under certain conditions on the coefficients σ and μ. However, as we want to study the consequences of the existence of boundaries we are not going into any detail about this.

For our analysis we restrict the process to an interval with at least one finite boundary, for simplicity of notation we consider the half-line $[0,\infty)$. A thorough study including an explicit formula for diffusions with reflecting boundary conditions on a finite interval $[0,a]$, for $a \in (0,\infty)$, can be found in [Kruk et al. 2007]. We are going to derive a generalization to this formula for time-dependent intervals in Chapter 6.

In our analysis we restrict ourself to the case of *instantaneous reflection*, in which at attaining the boundary the process returns to the interior in a continuous fashion in a way such that the measure of the time spent on the boundary is almost surely equal to zero. This implies directly that the values of μ and σ at the boundary don't matter. Some theory concerning other reflectional behavior, namely *absorption*, *delayed reflection* and *partial reflection*, can be found in A. V. Skorokhod's paper [Skorokhod 1961] and references found there.

We begin our examination by noting that equation (3.1) remains valid in the interior of the interval, as we want the boundary not to have any influence on the process in the interior of the interval. Thus, any change to equation (3.1) can only differ from zero on the boundary. The idea, we are going to follow, to force X to be non-negative is to introduce an additional term to equation (3.1), that grows whenever X threatens to become negative. Thus, we consider the Skorokhod problem:

Given diffusion coefficients μ and σ find a continuous X such that

$$\begin{aligned} dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dW_t + d\lambda_t \text{ for } t \geq 0, \\ X_t &\geq 0, \end{aligned} \tag{3.2}$$

where we assume $\lambda = (\lambda_t)_{t\geq 0}$ to be a continuous monotone function increasing only at the zeros of X. Skorokhod called the term λ *reflection function*. However, as we are going to see that λ is the local time of X at 0, we are going to call it accordingly.

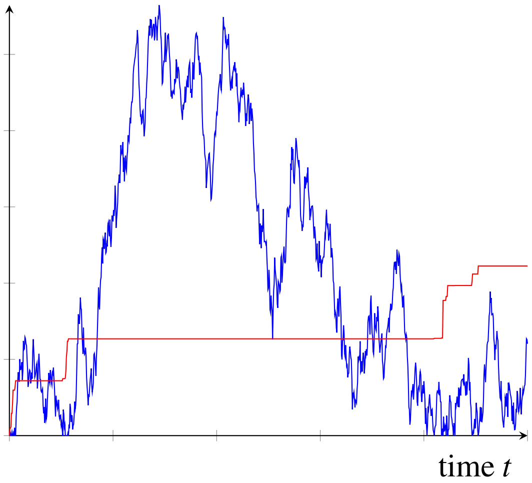

Figure 3.1.: The picture shows a reflected standard Brownian motion —— reflected at 0 and its local time —— at 0.

3.1. Uniqueness of reflected Brownian motion

The occurrence of two unknown functions in (3.2) seems to be rather inconvenient, luckily one can show that nevertheless the solution is unique (see Theorem 3.1) assuming that the coefficients μ and σ, defined on $[0,\infty)\times[0,\infty)$, are continuous and satisfy the *Lipschitz condition* in the second coordinate with a *Lipschitz constant* C, i.e.,

$$|\mu(t,x)-\mu(t,y)|\leq C|x-y| \text{ and } |\sigma(t,x)-\sigma(t,y)|\leq C|x-y| \quad \text{for } x,y\geq 0 \text{ and } t\geq 0. \tag{3.3}$$

Theorem 3.1 (Uniqueness)**.** *Suppose the coefficients μ and σ are Lipschitz-continuous, then* (3.2) *has at most one solution, more precisely any two solutions of* (3.2) *are indistinguishable.*

Proof. For the complete proof we refer to A. V. Skorokhod's paper [SKOROKHOD 1961] and restrict ourself to the special case in which $\mu = 0$ and $\sigma = 1$, i.e. X is a reflected standard Brownian motion.
So, let

$$X_t = X_0 + W_t + \lambda_t \text{ and } \tilde{X}_t = X_0 + W_t + \tilde{\lambda}_t$$

be two solutions to (3.2). In our simple setting we get directly $\tilde{X}_t - X_t = \tilde{\lambda}_t - \lambda_t$ and we can conclude that, as long as $\tilde{X} - X > 0$, $\tilde{\lambda}$ is non-increasing, since $\tilde{X} > X \geq 0$. In particular this implies that $\tilde{\lambda} - \lambda$ is non-increasing as long as it is positive. Analogously, we can establish that $\tilde{\lambda} - \lambda$ is non-decreasing as long as it is negative. As the difference is continuous and equal to zero at $t = 0$ this implies that $X = \tilde{X}$ and $\lambda = \tilde{\lambda}$ almost surely. □

3.2. Existence of reflected Brownian motion

After having established the uniqueness we turn towards the existence of the solution. We follow McKean's construction published in [MCKEAN 1963] and later on presented in more detail in [MCKEAN 1969] and start once more by considering the case of a reflected standard Brownian motion, i.e. $\mu = 0$ and $\sigma = 1$. Using Lévy's approach for a standard Brownian motion W starting at 0 we define

$$\lambda_t^{(0,1)} := -\min_{s\leq t} W_s^{+\,\mathrm{a}}$$

We can see that $\lambda^{(0,1)}$ increases only at the zeros of X and its definition ensures that X is non-negative, thus we have found the solution to (3.2) for $\mu = 0$ and $\sigma = 1$.
Before continuing we take a closer look at the distribution of $X_t = W_t + \lambda_t^{(0,1)}$.

$$\begin{aligned}
P(X_t < z) &= P(W_t - \min_{s \le t} W_s < z) \\
&= P(\max_{s \le t} W_s - W_t < z) \\
&=^{\text{b}} \int_0^\infty \int_{x-z}^x \sqrt{\frac{2}{\pi t^3}} (2x-y) \exp\left(-{(2x-y)^2}/{2t}\right) dy dx \\
&= \int_0^\infty \sqrt{\frac{2}{\pi t}} \left(\exp\left(-{x^2}/{2t}\right) - \exp\left(-{(x+z)^2}/{2t}\right) \right) dx \\
&= \int_0^z \sqrt{\frac{2}{\pi t}} \exp\left(-{x^2}/{2t}\right) dx \\
&= P(|W_t| < z)
\end{aligned}$$

As mentioned before, we can show that $\lambda_t^{(0,1)}$ is the local time of X at 0:

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \lambda \{ s \in [0,t]; X_s < \varepsilon \}^{\text{c}}$$

To do that we consider the expected squared difference of both terms, without taking the limit $\varepsilon \to 0$,

$$\mathbb{E}\left(\left| - \min_{s \le t} W_s - \frac{1}{2\varepsilon} \lambda \{ s \in [0,t]; X_s < \varepsilon \} \right|^2 \right). \tag{3.4}$$

We apply the Binomial Formula $(a+b)^2 = a^2 + 2ab + b^2$ to (3.4) and handle the three occuring terms $A := \mathbb{E}|\max_{s \le t} W_s|^2$, $-2B := -2\mathbb{E}\left(\max_{s \le t} W_s \cdot \frac{1}{2\varepsilon} \lambda \{ s \in [0,t]; X_s < \varepsilon \}\right)$ and $C := \mathbb{E}|\frac{1}{2\varepsilon} \lambda \{ s \in [0,t]; X_s < \varepsilon \}|^2$ separately:

$$A = \int_0^\infty \sqrt{\frac{2}{\pi t}} x^2 \exp(-{x^2}/{2t}) dx = t$$

$$\begin{aligned}
B &= \mathbb{E}\left(\max_{r \le t} W_r \frac{1}{2\varepsilon} \lambda \{ s \in [0,t]; W_s > \max_{r \le s} W_r - \varepsilon \} \right) \\
&= \frac{1}{2\varepsilon} \mathbb{E}\left(\max_{r \le t} W_r \int_0^t 1_{\{W_s > \max_{r \le s} W_r - \varepsilon\}}(s) ds \right) \\
&= \frac{1}{2\varepsilon} \int_0^t \mathbb{E}\left(\max_{r \le t} W_r \cdot 1_{\{W_s > \max_{r \le s} W_r - \varepsilon\}}(s) \right) ds \\
&\ge \frac{1}{2\varepsilon} \int_0^t \mathbb{E}\left(\left(W_s + \max_{r \le t-s} (W_{s+r} - W_s) \right) \cdot 1_{\{X_s > \max_{r \le s} W_r - \varepsilon\}}(s) \right) ds \\
&= \frac{1}{2\varepsilon} \int_0^t \int_0^\infty \int_{x-\varepsilon}^x \left(x + \sqrt{\frac{2(t-s)}{\pi}} \right) \cdot \sqrt{\frac{2}{\pi s^3}} (2x-y) \exp\left(-{(2x-y)^2}/{2s}\right) dy dx ds.
\end{aligned}$$

[a] x^+ denotes the positive part $x \wedge 0$ of x.
[b] For the joint distribution of W_t and $\max_{s \le t} W_s$ see (2.3)
[c] Here λ denotes the Lebesgue-measure.

Now we let $\varepsilon \to 0$, applying the Fundamental Theorem of Calculus we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0} B &= \frac{1}{2}\int_0^t \int_0^\infty \left(x+\sqrt{\frac{2(t-s)}{\pi}}\right)\cdot\sqrt{\frac{2}{\pi s^3}}x\exp\left(-x^2/2s\right)dxds \\
&= \frac{1}{2}\int_0^t \frac{1}{s}\underbrace{\int_0^\infty \sqrt{\frac{2}{\pi s}}x^2\exp\left(-x^2/2s\right)dx}_{=s}ds + \frac{1}{2}\int_0^t \sqrt{\frac{(t-s)}{\pi^2 s^3}}\underbrace{\int_0^\infty x\exp\left(-x^2/2s\right)dx}_{\left[s\exp\left(-x^2/2s\right)\right]_{x=0}^\infty}ds \\
&= \frac{1}{2}\int_0^t ds + \frac{1}{\pi}\int_0^t (t-s)^{1/2}s^{-1/2}ds \\
&= \frac{t}{2}+\frac{t}{\pi}\cdot\int_0^1 (1-\sigma)^{1/2}\sigma^{-1/2}d\sigma \\
&= t.
\end{aligned}$$

$$\begin{aligned}
C &= \frac{1}{(2\varepsilon)^2}\mathbb{E}\left(\left|\int_0^t 1_{X_s<\varepsilon}ds\right|^2\right) \\
&= \frac{1}{(2\varepsilon)^2}\mathbb{E}\left(\int_0^t\int_0^t 1_{\{X_s<\varepsilon\}}\cdot 1_{\{X_{\tilde{s}}<\varepsilon\}}d\tilde{s}ds\right) \\
&= \frac{1}{(2\varepsilon)^2}2\int_0^t\int_0^s P(X_{\tilde{s}}<\varepsilon, W_s<\varepsilon)d\tilde{s}ds \\
&= \frac{1}{2\varepsilon^2}\int_0^t\int_0^s\int_0^\varepsilon 2\sqrt{\frac{1}{2\pi\tilde{s}}}\exp\left(-x^2/2\tilde{s}\right) \\
&\qquad \cdot\int_0^{\varepsilon-x}\sqrt{\frac{1}{2\pi(s-\tilde{s})}}\left(\exp\left(-(x-y)^2/2(s-\tilde{s})\right)+\exp\left(-(x+y)^2/2(s-\tilde{s})\right)\right)dydxd\tilde{s}ds
\end{aligned}$$

Now, we let $\varepsilon \to 0$, applying the Fundamental Theorem of Calculus we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0} C &= \frac{1}{2}\int_0^t\int_0^s 2\sqrt{\frac{1}{2\pi\tilde{s}}}\sqrt{\frac{1}{2\pi(s-\tilde{s})}}(1+1)d\tilde{s}ds \\
&= \frac{1}{\pi}\int_0^t\int_0^s\sqrt{\frac{1}{\tilde{s}(s-\tilde{s})}}d\tilde{s}ds \\
&= t.
\end{aligned}$$

This gives us

$$\mathbb{E}\left|-\min_{s\le t}W_s-\lambda_t^{(0,1)}\right|^2 \le^{\mathrm{d}} \lim_{\varepsilon\to 0}\mathbb{E}\left|-\min_{s\le t}W_s-\frac{1}{2\varepsilon}\lambda\{s\in[0,t]; X_s<\varepsilon\}\right|^2 \le 0.$$

Thus, we have shown the proclaimed identity of $\lambda_t^{(0,1)} = -\min_{s\le t}W_s$ and the local time $\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\lambda\{s\in[0,t]; X_s<\varepsilon\}$ of X.
Next, we want to generalize obtained existence to coefficients depending on the value of X at

[d] We can e.g. apply Fatou's Lemma (see A.1).

a certain time, to obtain this we follow H. P. McKean's construction [McKEAN 1963] respectively [McKEAN 1969] using a **time substitution** to go over to arbitrary piecewise continuous diffusion coefficients $\sigma > 0$. We set

$$f(t) := \int_0^t {ds}/{\sigma^2(X_s)}$$

and use the inverse f^{-1} to define

$$Y := X \circ f^{-1} \text{ and } \lambda^{(0,\sigma^2)} := \lambda^{(0,1)} \circ f^{-1}$$

As we will see $(Y, \lambda^{(0,\sigma^2)})$ solve (3.2) and we can see directly that $\lambda^{(0,\sigma^2)}$ is the local time of Y:

$$\begin{aligned}
\lambda_t^{(0,\sigma^2)} &= \sigma^2(0) \cdot \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \lambda\{s \in [0,t]; Y_s < \varepsilon\} \\
&= \sigma^2(0) \cdot \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{[0,\varepsilon)}(Y_s) ds \\
&= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^{f^{-1}(t)} {\sigma^2(0)}/{\sigma^2(X_{s'})} \cdot 1_{[0,\varepsilon)}(X_{s'}) ds' \\
&\overset{\text{e}}{=} \lambda_{f^{-1}(t)}^{(0,1)}
\end{aligned}$$

To show that $Y = X \circ f^{-1} = W \circ f^{-1} + \lambda \circ f^{-1}$ satisfies (3.2) with $\mu = 0$ we apply the following theorem by Doob to $W \circ f^{-1}$ [DOOB 1953, page 449]:

Theorem 3.2. *Let $(X_t)_{t \in [0,T]}$ be a square-integrable martingale with the following properties.*

1. *Almost all sample paths of X are continuous in $[0,T]$.*
2. *There is a non-negative function $\sigma^2 : [0,T] \times \Omega \to (0,\infty)$, measurable with respect to the Borel σ-algebra on $[0,T] \times [0,\infty)$, such that, for each $t \in [0,T]$, $\sigma^2(t,\cdot)$ is measurable with respect to $\mathscr{F}_t$ and such that, for each $\tilde{t} < t \in [0,T]$,*

$$\mathbb{E}(|X_t - X_{\tilde{t}}|^2 | \mathscr{F}_{\tilde{t}}) = \mathbb{E}\left(\int_{\tilde{t}}^t |\sigma(s,\omega)|^2 ds | \mathscr{F}_{\tilde{t}} \right) \tag{3.5}$$

almost surely.

Then, there is a Brownian motion $(\tilde{W}_t)_{t \in [0,T]}$ such that for each $t \in [0,T]$,

$$X_t = X_0 + \int_0^t \sigma(s,\omega) d\tilde{W}_s \tag{3.6}$$

almost surely.

While $W \circ f^{-1}$ obviously possesses continuous sample paths, we have to make sure that it is in fact a martingale and that (3.5) holds. The first part is easily established noting that as $\{W_t; t \in [0,T]\}$ is uniformly integrable (defined in A.2) we can apply Doob's Optional-Sampling Theorem for uniformly integrable martingales (see A.3) and obtain that $W \circ f^{-1}$ is a martingale.

[e] ${\sigma^2(0)}/{\sigma^2(X_t)} \cdot 1_{[0,\varepsilon)}(X_t) \to 1_{[0,\varepsilon)}(X_t)$ as $\varepsilon \to 0$ since σ is continuous.

To show that (3.5) holds we apply Doob's Optional-Sampling Theorem for uniformly integrable martingales to $(W_t^2 - t)_{t\geq 0}$. Using the hereby obtained martingale property we get:

$$\mathbb{E}\big(|W_{f^{-1}(t)} - W_{f^{-1}(\tilde{t})}|^2 \big| \mathscr{F}_{f^{-1}(\tilde{t})}\big) = \mathbb{E}\big(f^{-1}(t) - f^{-1}(\tilde{t}) \big| \mathscr{F}_{f^{-1}(\tilde{t})}\big)$$
$$= \mathbb{E}\Big(\int_{\tilde{t}}^{t} \sigma^2(X_s)ds \Big| \mathscr{F}_{f^{-1}(\tilde{t})}\Big).$$

As $\sigma \circ X$ fulfills the measurability conditions and W is square-integrable Theorem 3.2 yields that

$$W_{f^{-1}(t)} = a + \int_0^t \sigma(X_s)d\tilde{W}_s$$

for some Brownian motion $\tilde{W}$, i.e. X solves (3.2) with $\mu = 0$.
To obtain an arbitrary piecewise continuous drift coefficient μ, we **change the scale** defining $l : a \mapsto l(a)$ implicitly by

$$a = \int_0^l \exp\left(-2\int_0^b {}^{\mu(y)}\!/_{\sigma^2(y)} dy\right) db.$$

Using this new scale we rescale the previously obtained solution to $dX_t = \sigma^*(X_t)dW_t + d\lambda_t$, where we set $\sigma^*(x) := \sigma(l(x)) \cdot \exp(-2\int_0^{l(x)} {}^{\mu(y)}\!/_{\sigma(y)} dy)$, and consider the rescaled version $Z = l(X)$. Itô's formula yields

$$dZ_t = l'(X_t)dX_t + {}^1\!/_2 \cdot l''(X_t)d[X]_t. \tag{3.7}$$

From the implicit definition of l we obtain by applying the Implicit Function Theorem (see A.4) the derivatives:

$$l'(x) = -\frac{1}{-\exp\left(-2\int_0^{l(x)} {}^{\mu(y)}\!/_{\sigma^2(y)} dy\right)} = \exp\left(2\int_0^{l(x)} {}^{\mu(y)}\!/_{\sigma^2(y)} dy\right)$$
$$l''(x) = l'(x) \cdot 2\frac{\mu(l(x))}{\sigma^2(l(x))} \cdot l'(x) = 2\frac{\mu(l(x))}{\sigma^2(l(x))} \cdot (l'(x))^2$$

Plugging this into (3.7) and using $d[W,\lambda]_t = d[\lambda]_t = 0$ as well as $d[W]_t = dt$ we get

$$dZ_t = l'(X_t) \cdot \underbrace{(\sigma^*(X_t)dW_t + d\lambda_t)}_{=dX_t} + l''(X_t)(\sigma^*(X_t))^2 \cdot \underbrace{d[X]_t}_{=dt}$$
$$= \underbrace{l''(X_t)(\sigma^*(X_t))^2}_{=\mu(Z_t)} dt + \underbrace{l'(X_t) \cdot \sigma^*(X_t)}_{\sigma(Z_t)} dW_t + l'(X_t) \cdot d\lambda_t$$
$$= \mu(Z_t)dt + \sigma(Z_t)dW_t + d\lambda_t$$

Where in the last step we use that $d\lambda_t$ is only non-vanishing on $\{t \in [0,T]; X_t = 0\}$ and $l'(0) = 1$.

A short calculation gives us that λ is the local time of Z:

$$
\begin{aligned}
&\sigma^2(0)\cdot\lim_{\varepsilon'\to 0}\frac{1}{2\varepsilon'}\lambda\{s\in[0,t];Z_s<\varepsilon'\}\\
=&\sigma^2(0)\cdot\lim_{\varepsilon'\to 0}\frac{1}{2\varepsilon'}\int_0^t 1_{[0,\varepsilon')}(l(X_s))ds\\
=&\sigma^2(0)\cdot\lim_{\varepsilon'\to 0}\frac{1}{2\varepsilon'}\int_0^t 1_{[0,l^{-1}(\varepsilon'))}(X_s)ds\\
=&\sigma^2(0)\cdot\lim_{\varepsilon\to 0}\frac{1}{2l(\varepsilon)}\int_0^t 1_{[0,\varepsilon)}(X_s)ds\\
=&\sigma^2(0)\cdot\lim_{\varepsilon\to 0}\frac{1}{2\cdot(l(0)+\varepsilon\cdot l'(0)+O(\varepsilon^2))}\int_0^t 1_{[0,\varepsilon)}(X_s)ds\\
=&\sigma^2(0)\cdot\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t 1_{[0,\varepsilon)}(X_s)ds\\
=&\lambda_t
\end{aligned}
$$

We note that for existence in the case of time-independent coefficients μ and σ, piecewise continuity is sufficient, while for uniqueness we need in general that both coefficients are Lipschitz continuous in the second coordinate.
We summarize our previous results by quoting A. V. Skorokhod's "Fundamental Result" and notice the additional condition necessary for existence of a solution given time-dependent coefficients.

Theorem 3.3. *Let the coefficients* $\mu:[0,\infty)\times[0,\infty)\to\mathbb{R}$ *and* $\sigma:[0,\infty)\times[0,\infty)\to(0,\infty)$ *be continuous and Lipschitz continuous in the second coordinate. Moreover, assume that there exists a* $\delta>0$ *such that the partial derivatives* $\frac{\partial\sigma}{\partial t}$ *are defined, continuous and Lipschitz continuous in the second coordinate on* $[0,T]\times[0,\delta]$.
Then for any non-negative, square-integrable variable x independent of the Brownian motion W there exists a unique continuous Markov process X that solves (3.2) *with* $X_0=x$.

Proof. The proof can be found in [SKOROKHOD 1961]. □

4. Multi-dimensional diffusions with reflecting boundaries

In Chapter 3 we have established a comprehensive theory for the one-dimensional case. Next we turn towards the multi-dimensional case. Here, we have to make some additional assumptions to obtain existence and uniqueness. In fact several different conditions are known to lead to the desired result. In the following we devote an individual section to each one of them. Before we turn towards these assumptions we state the problem in the most general form and give some insight into the requirements on a sufficient condition.
We consider a domain D in $\mathbb{R}^d$ (a domain is an open set that equals the inner of its closure), we denote its closure by $\overline{D}$ and its boundary by ∂D. Moreover, we introduce a notation for normal vectors on the surface: For $x \in \partial D$ we define the set of inward normal unit vector $\mathscr{N}_x$ by

$$\begin{aligned} \mathscr{N}_x &= \bigcup_{r>0} \mathscr{N}_{x,r}, \\ \mathscr{N}_{x,r} &= \{\mathbf{n} \in \mathbb{R}^d; |\mathbf{n}| = 1, B_r(x - r\mathbf{n}) \cap D = \emptyset\}, \end{aligned} \tag{4.1}$$

where $B_r(y)$ is the open ball around $y \in \mathbb{R}^d$ with radius $r > 0$. Correspondingly we denote the set of supporting hyperplanes $\mathscr{H}_x$ at $x \in \partial D$.
Throughout the chapter we will consider continuous functions f mapping from $[0,\infty)$ to either $\mathbb{R}^d$ or the subset $\overline{D}$, we denote the set of such functions by $C([0,\infty),\mathbb{R}^d)$ and $C([0,\infty),\overline{D})$, respectively. The same applies for càdlàg functions, we denote the set with $D([0,\infty),\mathbb{R}^d)$ and $D([0,\infty),\overline{D})$, respectively. For a function $g \in D([0,\infty),\mathbb{R}^d)$ we denote its total variation until $t \in [0,\infty)$ by $|g|(t) := \sup_{\text{partitions}(t_k)_{k=\{1,\dots,n\}}} \sum_{k=1}^n (t_k - t_{k-1}) \cdot |g(t_k) - g(t_{k-1})|$.
To state the deterministic problem we introduce the notation of *associated* functions:

Definition 4.1. *Let $f \in D([0,\infty),\overline{D})$. A function $h \in D([0,\infty),\mathbb{R}^d)$ is said to be associated with f if the following conditions are satisfied:*

1. *h is of bounded variation.*
2. *The set $\{t \in [0,\infty); f(t) \in D\}$ has $d|h|$-measure zero.*
3. *In the representation*

$$h(t) = \int_0^t n(s) d|h|(s)$$

the vector $n(s)$ is a normal vector at $f(s)$ for almost all s with respect to the measure $d|h|$.

Now we are equipped to state the deterministic Skorokhod problem:
Given a function $g \in D([0,\infty),\mathbb{R}^d)$ starting in $g(0) \in \overline{D}$, find a solution (f,h) of

$$f = g + h, \tag{4.2}$$

where f is in $D([0,\infty),\overline{D})$ and h is associated to f. The total variation $|h|$ is called the local time of the solution. For g such that we can find a solution (f,h) we define the (possibly multi-valued) Skorokhod map $g \mapsto f$.
The idea we are going to follow to find a solution to (4.2) is to start by considering a step function g. As soon as, for some $t \in [0,\infty)$, $g(t)$ is not in $\overline{D}$ anymore, we project it back into $\overline{D}$. This projection has to be chosen such that the obtained function h is associated to f. Thus, we are looking for conditions ensuring the existence of such a projection.
Using successively smoother partitions we can carry this result over to arbitrary càdlàg functions.
This result can be used to obtain a stochastic version of (4.2) and the solution to the Skorokhod SDE (4.13).

4.1. SDEs with reflecting boundaries in convex regions

We will see that the convexity of D is a sufficient condition for the existence and uniqueness of the solution to the Skorokhod SDE (4.13). The thorough theory about diffusions in convex regions is due to H. Tanaka [Tanaka 1979].
As laid out previously we start by considering deterministic step functions.

Lemma 4.2. *Suppose D is convex. If g is a step function starting in $g(0) \in \overline{D}$, then a solution of (4.2) exists.*

Proof. We define $f(t) := g(t)$ for $0 \le t \le \tau_1 := \inf\{t > 0; g(t) \notin \overline{D}\}$ and $f(\tau_1) := \overline{g(t)}$, where $\mathbb{R}^d \to \overline{D}, x \mapsto \bar{x} := \operatorname{argmin}_{y\in\overline{D}}|x-y|$ is well-defined because of the convexity of $\overline{D}$. Using this construction we have defined the solution on $[0,\tau_1]$ and continue by induction. Thus, we assume that we have established the solution on $[0,\tau_n]$, where we continue $\tau_n := \inf\{t > \tau_{n-1}; g(t) + h(\tau_{n-1}) \notin \overline{D}\}$ and set

$$f(t) := \begin{cases} g(t)+h(\tau_n) & \text{for} \quad \tau_n < t < \tau_{n+1} \\ \overline{g(t)+h(\tau_n)} & \text{for} \quad t = \tau_{n+1}. \end{cases}$$

Then, f solves (4.2) on $[0,\tau_{n+1}]$. The construction ensures additionally that h is associated with f. Since $\tau_n \to \infty$ as $n \to \infty$, we obtain a solution for $t \in [0,\infty)$. □

Before we carry this result over to arbitrary functions $g \in D([0,\infty),\mathbb{R}^d)$, we take a look at an existing solution and show that continuity of g carries over to f and h and obtain the uniqueness of the solution. To do that we need the following lemma:

Lemma 4.3. *Suppose D is convex. Let g, $\tilde{g} \in D([0,\infty),\mathbb{R}^d)$ starting in $g(0)$, $\tilde{g}(0) \in \overline{D}$, and f, $\tilde{f}$ be solutions of* (4.2) *with $f = g+h$ and $\tilde{f} = \tilde{g}+\tilde{h}$, respectively. Then we have*

$$|f(t)-\tilde{f}(t)|^2 \le |g(t)-\tilde{g}(t)|^2 + 2\int_0^t \langle g(t)-\tilde{g}(t)-g(s)+\tilde{g}(s), h(ds)-\tilde{h}(ds)\rangle, \tag{4.3}$$

$$|f(t)-f(\tilde{t})|^2 \le |g(t)-g(\tilde{t})|^2 + 2\int_{\tilde{t}}^t \langle g(t)-g(s), h(ds)\rangle \tag{4.4}$$

for $0 \le \tilde{t} \le t < \infty$.

Proof. The proof is a straightforward calculation and can be found in [Tanaka 1979, Lemma 2.2]. □

Now, we use (4.3) to establish the uniqueness, by considering two solutions f, $\tilde{f}$ of (4.2). Setting $g = \tilde{g}$ in (4.3) we obtain directly $|f(t) - \tilde{f}(t)|^2 \leq 0$ and thus:

Corollary 4.4. *If D is convex, then* (4.2) *has at most one solution.*

To examine the consequence of the continuity of g we lessen (4.4) to

$$|f(t) - f(\tilde{t})|^2 \leq |g(t) - g(\tilde{t})|^2 + 2\int_{\tilde{t}}^{t} |g(t) - g(s)| \cdot |h|(t) ds$$

and get directly:

Corollary 4.5. *Suppose D is convex. If g is continuous, then so is the solution of (4.2).*

Lemma 4.6. *Suppose D is convex. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $D([0,\infty),\mathbb{R}^d)$ such that for each $n \in \mathbb{N}$ the equation $f_n = g_n + h_n$ has a solution according to* (4.2) *on* $[0,T]$. *If g_n converges uniformly on $[0,T]$ to some $g \in C([0,\infty),\mathbb{R}^d)$ as $n \to \infty$ and if $(|h_n|(T))_{n\in\mathbb{N}}$ is bounded, then $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[0,T]$ as $n \to \infty$ to the solution of f of $f = g + h$ on $[0,T]$.*

Proof. We apply (4.3) to f_n and f_m for $n,m \in \mathbb{N}$ and use the boundedness of $(|h_n|(T))_{n\in\mathbb{N}}$ to obtain the uniform convergence of $(f_n)_{n\in\mathbb{N}}$ on $[0,T]$, this implies the uniform convergence for $(h_n)_{n\in\mathbb{N}}$ on $[0,T]$. Analogously, taking the limit in the inequality obtained by applying (4.4) to f_n yields the continuity of f. We only have to show that f is a solution of (4.2), i.e. that h is associated to f. From the bounded variation of $(h_n)_{n\in\mathbb{N}}$ we obtain directly the bounded variation of the limit h and the second condition can be seen easily. The last condition is verified via a technical alternative condition. For details see [TANAKA 1979, Lemma 2.5]. □

To show the existence and uniqueness for a convex region some additional condition is needed, we begin by proving the existence under a fairly restrictive condition and show later on that a local version of this condition suffices and that we can find a sufficient condition easy to understand.

Lemma 4.7. *Suppose D is convex and there exists a unit vector v and a constant $c > 0$ such that a solution $\langle v, \mathbf{n}\rangle \geq c$ for any $\mathbf{n} \in \bigcup_{y\in\partial D} \mathcal{N}_y(D)$. For any $g \in C([0,\infty),\mathbb{R}^d)$ there exists a solution (f,h) of* (4.2) *and for each* $0 \leq \tilde{t} \leq t$

$$|f(t) - f(\tilde{t})| \leq K\Delta_{\tilde{t},t}, \tag{4.5}$$

$$|h|(t) - |h|(\tilde{t}) \leq K'\Delta_{\tilde{t},t}, \tag{4.6}$$

where K and K′ are constants depending only upon the constant c and the variability of g $\Delta_{\tilde{t},t} = \sup_{\tilde{t}\leq r<s\leq t} |g(s) - g(r)|$.

Proof. Applying Lemma 4.2 we obtain the solution f_n of $f_n = g_n + h_n$ for the discretization $g_n(t) := g(\frac{k}{n})$ for $\frac{k-1}{n} \leq t \leq \frac{k}{n}$. We define the discretized version of $\Delta_{\tilde{t},t}$ and $K_{\tilde{t},t}$:

$$\Delta_{n,\tilde{t},t} := \sup_{\tilde{t}\leq r<s\leq t} |g_n(s) - g_n(r)|,$$

$$K_{n,\tilde{t},t} := |h_n|(t) - |h_n|(\tilde{t}),$$

and start by obtaining estimates analog to (4.5) and (4.6) by considering (4.4):

$$\begin{aligned}|f_n(t)-f_n(\tilde{t})|^2 &\le |g_n(t)-g_n(\tilde{t})|^2 + 2\int_{\tilde{t}}^t \langle g_n(t)-g_n(s), h_n(ds)\rangle \\ &\le \Delta^2_{n,\tilde{t},t} + 2K_{n,\tilde{t},t}\Delta_{n,\tilde{t},t} \\ &\le \Delta^2_{n,\tilde{t},t} + \varepsilon^2 K_{n,\tilde{t},t}{}^2 + \Delta^2_{n,\tilde{t},t}/\varepsilon^2 \text{ for arbitrary } \varepsilon > 0.\end{aligned}$$

It is left to estimate $K_{n,\tilde{t},t}$. We note that establishing a bound will allow us as well to apply Lemma 4.6. We obtain the estimate by considering

$$\begin{aligned}\langle v, f_n(t)-f_n(\tilde{t})\rangle &= \langle v, g_n(t)-g_n(\tilde{t})\rangle + \langle v, h_n(t)-h_n(\tilde{t})\rangle \\ &\ge \langle v, g_n(t)-g_n(\tilde{t})\rangle + cK_{n,\tilde{t},t},\end{aligned}$$

v as in the assumption, that is, $K_{n,\tilde{t},t} \le \frac{|f_n(t)-f_n(\tilde{t})|+\Delta_{n,\tilde{t},t}}{c}$.
Put together we have $|f_n(t)-f_n(\tilde{t})| \le (1+\frac{1}{\varepsilon}+\frac{\varepsilon}{c})\Delta_{\tilde{t},t} + \frac{\varepsilon}{c}|f_n(t)-f_n(\tilde{t})|$, and therefore

$$|f_n(t)-f_n(\tilde{t})| \le K\Delta_{n,\tilde{t},t} \text{ and } K_{n,\tilde{t},t} \le K'\Delta_{n,\tilde{t},t}, \tag{4.7}$$

where K is the minimum of $\left(1+\frac{1}{\varepsilon}+\frac{\varepsilon}{c}\right)\cdot\left(1-\frac{\varepsilon}{c}\right)^{-1}$ for ε ranging over the interval $(0,c)$ and $K' = {}^{(1+K)}/_{c}$.
As we have in particular that $(|h_n|(T))_{n\in\mathbb{N}}$ is bounded, we obtain by Lemma 4.6 that f_n converges uniformly on compacts to the solution f of (4.2). Taking the limit in (4.7) we obtain directly (4.5) and (4.6). □

Condition A. *There exist* $\varepsilon > 0$ *and* $\delta > 0$ *such that for any* $x \in \partial D$ *we can find an open ball* $B_\varepsilon(x_0) \subset D$ *satisfying* $|x - x_0| \le \delta$.

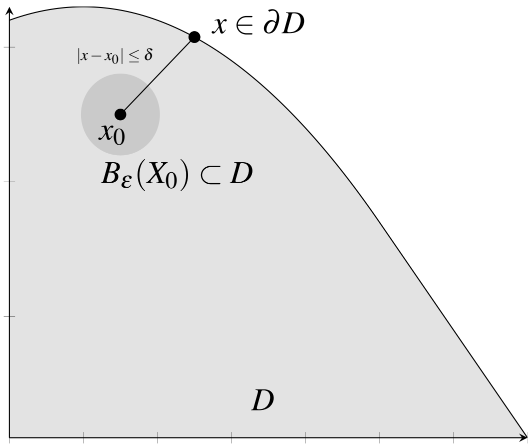

Figure 4.1.: Example for a set satisfying Condition A.

For bounded domains D (set $\delta := \sup_{x\in\partial D}|x-x_0|$ for some inner point x_0) and $d=2$ the condition is always satisfied.
We mentioned before that if D satisfies Condition A, then it satisfies locally the condition necessary for Lemma 4.7, to see this we put for any $x \in \partial D$

$$D_x := \bigcap_{y\in\partial D\cap\overline{B_{\varepsilon/2}}(x)} \; \bigcap_{H\in\mathscr{H}_y} H(D),$$

where $H(D)$ denotes the open half-space bounded by a supporting hyperplane H and containing D (as a consequence of the convexity D). Then D_x is a convex cone domain satisfying the condition of Lemma 4.7 with

$$v = {}^{(x_0-x)}/_{|x_0-x|} \text{ and } c = {}^{\varepsilon}/_{2\delta}$$

where $x_0 \in D$, $\varepsilon > 0$ and $\delta > 0$ are as in Condition A.
Next, we show how to utilize this to prove that Condition A is sufficient for the existence of a unique solution to the deterministic Skorokhod problem.

Theorem 4.8. *Suppose D is convex.*

1. *If D satisfies Condition A, then there exists for any $g \in C([0,\infty),\mathbb{R}^d)$ a unique solution of* (4.2)*, and the solution f depends continuously upon g with respect to the compact topology.*
2. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence on $C([0,\infty),\mathbb{R}^d)$ such that $f_n = g_n + h_n$ has a solution for each $n \in \mathbb{N}$. If $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ converge to f and g on compacts as $n \to \infty$, respectively, then f is a solution of* (4.2).

Proof. As done in Lemma 4.2 we prove the statement by induction. Again, until $\tau_1 := \inf\{t > 0; g(t) \notin \overline{D}\}$ we can set f_1 to be g. Assuming that the solution f_{n-1} of (4.2) has been established on $[0,\tau_{n-1}]$, τ_{n-1} defined as in (4.9), i.e. in particular $f_{n-1}(\tau_{n-1}) \in \partial D$, we turn towards extending it beyond τ_{n-1}. To do so we consider $\overline{D_{f_{n-1}(\tau_{n-1})}}$ for which there exists a solution $\tilde{f}_n = \tilde{g}_n + \tilde{h}_n$ for $\tilde{g}_n(t) := g(\tau_{n-1}+t)$ due to Lemma 4.7. We put

$$\tilde{\tau}_n := \inf\{t \geq \tau_{n-1}; |\tilde{f}_n(t-\tau_{n-1}) - \tilde{\tau}_n(0)| = {}^{\varepsilon}/_2\}, \tag{4.8}$$

$$\tau_n := \inf\{t \geq \tilde{\tau}_n; \tilde{f}_n(\tilde{\tau}_n - \tau_{n-1}) + g(t) - g(\tilde{\tau}_n) \notin \overline{D}\}. \tag{4.9}$$

As until $\tilde{\tau}_n$, $t \mapsto \tilde{f}_n(t-\tau_{n-1})$ is in $\overline{D_{f_{n-1}(\tau_{n-1})}}$ and thus in particular in $\overline{D}$ and as on $[\tilde{\tau}_n, \tau_n]$ the definition of τ_n ensures that $t \mapsto \tilde{f}_n(\tilde{\tau}_n - \tau_{n-1}) + g(t) - g(\tilde{\tau}_n)$ remains in $\overline{D}$, we can set

$$f_n(t) := \begin{cases} f_{n-1}(t) & \text{for} \quad 0 \leq t \leq \tau_{n-1}, \\ \tilde{f}_n(t-\tau_{n-1}) & \text{for} \quad \tau_{n-1} \leq t \leq \tilde{\tau}_n, \\ \tilde{f}_n(\tilde{\tau}_n - \tau_{n-1}) + g(t) - g(\tilde{\tau}_n) & \text{for} \quad \tilde{\tau}_n \leq t \leq \tau_n, \end{cases}$$

and we obtain the solution of (4.2) on $\overline{D}$ for $0 \leq t \leq \tau_n$. This gives us the solution for $t < \tau_\infty = \lim_{n\to\infty} \tau_n$. To obtain that $\tau_\infty = \infty$, we consider (4.5) yielding for $[\tau_{n-1}, \tilde{\tau}_n]$ with constant K depending only on $c = {}^{\varepsilon}/_{2\delta}$ that

$${}^{\varepsilon}/_{2K} \leq \Delta_{\tau_{n-1},\tilde{\tau}_n} \leq \Delta_{\tau_{n-1},\tau_n}.$$

Additionally we use that g is continuous to obtain for any time $T > 0$ a constant $d > 0$, s.t.

$$\Delta_T(h) := \max\{|g(t) - g(\tilde{t})|; 0 \leq \tilde{t} < t \leq T \text{ and } t - \tilde{t} < d\} < {}^{\varepsilon}/_{2K}.$$

Put together we have $\Delta_{\tau_n}(d) < \Delta_{\tilde{\tau}_n,\tau_n}$, in particular, $\tau_n - \tau_{n-1} > d$, thus $\tau_n > n \cdot d$ providing us with $\tau_\infty = \infty$. Putting together (4.5) and (4.6), respectively, for all intervals $[\tau_{n-1}, \tau_n]$ we obtain for $0 \leq \tilde{t} \leq t \leq T$

$$\begin{aligned} |f(t) - f(\tilde{t})| &\leq \left(\frac{T}{d} + 1\right) K \Delta_{\tilde{t},t}, \\ |h|(t) - |h|(\tilde{t}) &\leq \left(\frac{T}{d} + 1\right) K' \Delta_{\tilde{t},t}. \end{aligned} \tag{4.10}$$

We will only give a sketch of the proof of the second statement, the details can be found in [TANAKA 1979, Theorem 2.1]. We start of by noticing that for D satisfying Condition A we can use (4.10) to obtain that $(|h|(T))_{n\in\mathbb{N}}$ is bounded, note that this requires some argumentation on how to choose the constant d independent of n, and we can apply Lemma 4.6 to obtain $f_n \to f$ uniformly on $[0,T]$.
For a sequence converging on compacts use that for any $T > 0$ there exists $N > 0$ such that

$$\sup_{n\in\mathbb{N}} \max_{0\le t\le T} |f_n(t)| < N.$$

As $D_N := D \cap B_N(0)$ satisfies Condition A and f_n respectively f are the solutions of $f_n = g_n + h_n$ and $f = g + h$ for D_N on $[0,T]$ with $T > 0$ arbitrary. With our definition of D_N these are the solutions on D. □

Next, we want to carry our results over to stochastic processes. Additionally, we will obtain that we can drop Condition A when taking g from sample path of a continuous semimartingale. We start by introducing a complete probability space $(\Omega, \mathscr{A}, \mathbb{P})$ and a filtration $\mathscr{F} = (\mathscr{F}_t)_{t\ge 0}$ fulfilling the usual conditions.

Theorem 4.9. *Suppose D is convex. If $(Y_t)_{t\ge0}$ is an $\mathbb{R}^d$-valued semimartingale starting in $Y_0 \in \overline{D}$, then there exists a unique $\mathscr{F}$-adapted solution $X = (X_t)_{t\ge0}$ of*

$$X_t = Y_t + \Phi_t, \tag{4.11}$$

where a solution means a $\overline{D}$-valued process X and a process Φ, whose sample path are associated to the sample path of X almost surely.
Moreover, for $f \in C^2([0,\infty),\mathbb{R})$ with $f' \ge 0$ on $[0,\infty)$ and $0 \le s \le t$ we have

$$\begin{aligned} f(|X_t - X_s|^2) \le f(0) &+ 2\sum_i \int_s^t f'(|X_r - X_s|^2)(X_r^i - X_s^i)\,dY_r^i \\ &+ 2\sum_{i,j} \int_s^t f''(|X_r - X_s|^2)(X_r^i - X_s^i)(X_r^j - X_s^j)\,d[X^i, X^j]_r \\ &+ \sum_i \int_s^t f'(|X_r - X_s|^2)\,d[X^i]_r. \end{aligned} \tag{4.12}$$

Proof. The proof can be found in [TANAKA 1979, Theorem 3.1]. □

Now, we turn towards the result we are interested in mostly and consider the stochastic differential equation with reflection:

$$\begin{aligned} dX_t &= \mu(t,X_t)dt + \sigma(t,X_t)dW_t + d\Phi_t \\ X_0 &= x_0 \end{aligned} \tag{4.13}$$

where σ is an $\mathbb{R}^d \otimes \mathbb{R}^r$-valued function and μ is an $\mathbb{R}^d$-valued function on $[0,\infty)\times\overline{D}$, both functions are assumed to be Borel-measurable. Our task is to find an $\mathscr{F}$-adapted $\overline{D}$-valued process X under the condition that $\Phi(\omega)$ is associated to $X(\omega)$ for each $\omega \in \Omega$.

Theorem 4.10. *Suppose D is convex. If σ and μ are Lipschitz-continuous in the second coordinate and fulfill the linear growth conditions*

$$\|\mu(t,x)\| \le K(1+|x|^2)^{1/2} \text{ and } \|\sigma(t,x)\| \le K(1+|x|^2)^{1/2},$$

then there exists a (pathwise) unique $\mathscr{F}$-adapted solution of (4.13) for any $x_0 \in \overline{D}$.

Proof. To prove the uniqueness the following additional estimate close to (4.3) is needed, for details and how to derive this we refer to [TANAKA 1979, Remark 2.2, Theorem 4.1]:
Let X and $\tilde{X}$ be $\mathscr{F}$-adapted solutions of (4.13), then

$$\begin{aligned} |X_t - \tilde{X}_t|^2 \leq & \left| \int_0^t \sigma(s,X_s)dW_s - \int_0^t \sigma(s,\tilde{X}_s)dW_s \right|^2 \\ & +2\int_0^t \langle X_s - \tilde{X}_s, \mu(s,X) - \mu(s,\tilde{X}_s) \rangle ds \\ & +2\int_0^t \langle \int_s^t \sigma(r,X_r) - \sigma(r,\tilde{X}_r)dr, (\mu(s,X_s) - \mu(s,\tilde{X}_s))ds + d\Phi_s - d\tilde{\Phi}_s \rangle \end{aligned} \tag{4.14}$$

We notice that the third term has zero expectation and hence

$$\begin{aligned} \mathbb{E}|X_t - \tilde{X}_t|^2 \leq & \mathbb{E}\int_0^t \|\sigma(s,X_s) - \sigma(s,\tilde{X}_s)\|^2 ds \\ & +\mathbb{E}\int_0^t |X_s - \tilde{X}_s|^2 ds + \mathbb{E}\int_0^t |\mu(s,X_s) - \mu(s,\tilde{X}_s)|^2 ds \\ \leq & (2K^2+1)\int_0^t \mathbb{E}|X_s - \tilde{X}_s|^2 ds. \end{aligned} \tag{4.15}$$

Applying Gronwall's Lemma gives us $\mathbb{E}|X_t - \tilde{X}_t|^2 = 0$, i.e. the solution is unique.
To prove the existence we start by assuming that D is bounded enabling us to apply Theorem 4.8. Accordingly, we can define a sequence $(X^{(n)})_{n\in\mathbb{N}}$ of $\overline{D}$-valued processes by

$$\begin{aligned} X_t^{(0)} &= x_0, \\ X_t^{(n)} &= x_0 + \int_0^t \mu(s,X_s^{(n-1)})ds + \int_0^t \sigma(s,X_s^{(n-1)})dW_s + \Phi_t^{(n)}, \text{ for } n \geq 1. \end{aligned}$$

To apply Theorem 4.8 we have to show that

$$\left(\int_0^t \mu(s,X_s^{(n)})ds + \int_0^t \sigma(s,X_s^{(n)})dW_s \right)_{n\in\mathbb{N}}$$

converges uniformly on compacts almost surely. Our idea is to show that it is uniformly Cauchy, to do that we consider the difference of two consecutive elements of the sequence, we start by considering the finite variance part and the martingale part separately using the notation

$$\begin{aligned} V_t^{(n)} &:= \int_0^t \mu(s,X_s^{(n)}) - \mu(s,X_s^{(n-1)})ds, \\ M_t^{(n)} &:= \int_0^t \sigma(s,X_s^{(n)}) - \sigma(s,X_s^{(n-1)})dW_s. \end{aligned}$$

Then, we can obtain the following estimates for $T > 0$:

$$\begin{aligned} \mathbb{E}\left(\max_{t\leq T} \left(V_t^{(n)} \right)^2 \right) &\leq {}^{\mathrm{a}}K^2 \cdot T \cdot \mathbb{E}\Big(\int_0^t |X_T^{(n)} - X_T^{(n-1)}|^2 \Big) \\ \mathbb{E}\left(\max_{t\leq T} \left(M_t^{(n)} \right)^2 \right) &\leq {}^{\mathrm{b}}4\mathbb{E}\Big(\int_0^T \sigma(s,X_s^{(n)}) - \sigma(s,X_s^{(n-1)})dW_s \Big)^2 \\ &= 4\mathbb{E}\Big(\int_0^T \left(\sigma(s,X_s^{(n)}) - \sigma(s,X_s^{(n-1)}) \right)^2 ds \Big) \\ &\leq 4K^2\mathbb{E}\left(\int_0^T |X_s^{(n)} - X_s^{(n-1)}|^2 ds \right) \end{aligned}$$

The term $\mathbb{E}(\int_0^T |X_s^{(n)} - X_s^{(n-1)}|^2)$, occurring in both estimates, can be treated the same way as the two solutions in (4.15) and we can obtain

$$\mathbb{E}|X_T^{(n+1)} - X_T^{(n)}|^2 \leq (2K^2+1)\int_0^T \mathbb{E}|X_s^{(n)} - X_s^{(n-1)}|^2 ds. \tag{4.16}$$

Now, we are in the position to estimate the expected maximum over the squared distance between two consecutive elements of $(\int_0^t \mu(s, X_s^{(n)})ds + \int_0^t \sigma(s, X_s^{(n)})dW_s)_{n\in\mathbb{N}}$

$$\begin{aligned}
\mathbb{E}\max_{t\leq T}\left(V_t^{(n)} + M_t^{(n)}\right)^2 &\leq 2\mathbb{E}\left(\max_{t\leq T}\left(V_t^{(n)}\right)^2 + \max_{t\leq T}\left(M_t^{(n)}\right)^2\right) \\
&\leq 2K^2(T+4)\cdot\mathbb{E}\left(\int_0^T |X_s^{(n)} - X_s^{(n-1)}|^2\right)ds \\
&\leq {}^{c}2K^2(T+4)\cdot(2K^2+1)^n \int_0^T \cdots \int_0^{s_n} \mathbb{E}|X_{s_n}^{(1)} - x_0|^2 ds_{n+1}\cdots ds_1 \\
&\leq \underbrace{2K^2(T+4)\cdot\max_{s\leq T}\mathbb{E}|X_s^{(1)} - x|^2}_{=:\tilde{K}\text{ constant independent of } n}\cdot(2K^2+1)^n\cdot\frac{T^{n+1}}{(n+1)!}.
\end{aligned} \tag{4.17}$$

Now, we apply Markov's inequality (see A.7) to obtain an estimate for

$$\mathbb{P}\underbrace{\left(\max_{t\leq T}|V_t^{(n-1)} + M_t^{(n-1)}|^2 \geq 1/2^n\right)}_{=:A_n} \leq \hat{K}\cdot\frac{\bar{K}^n\cdot T^n}{n!},$$

for constants $\hat{K}$ and $\bar{K}$ independent of n. We can put these estimates together to obtain

$$\sum_{n=1}^{\infty}\mathbb{P}(A_n) \leq \hat{K}\cdot\exp(\bar{K}\cdot T) < \infty$$

thus, Borell-Cantelli's Lemma (see A.8) implies that, for $\mathbb{P}$-almost all $\omega\in\Omega$,

$$\left(\int_0^t \mu(s, X_s^{(n)})ds + \int_0^t \sigma(s, X_s^{(n)})dW_s\right)_{n\in\mathbb{N}}$$

is uniformly Cauchy and, thus, uniformly convergent almost surely.
The continuity of the Skorokhod map, given in Theorem 4.8, implies that $(X^{(n)})_{n\in\mathbb{N}}$ is as well uniformly convergent and the obtained limit process X solves (4.13). For the remaining part on how to carry this result over to unbounded D, we refer to [TANAKA 1979, Theorem 4.1]. The idea is to consider $D_n := \{x\in D; |x| < n\}$ with solution $X^{(n)}$ and show that the sequence of solutions can be used to define the limit by

$$X_t(\omega) := X_t^{(n)}(\omega)$$

for $t \leq \tau_n(\omega) := \inf\{t\geq 0; |X_t^{(n)}(\omega)| = n\}$. □

[a] We apply the Hölder inequality (see A.5) and utilize the Lipschitz continuity of μ.
[b] As $M^{(n)}$ is a martingale we can apply Doob's Maximal Inequality (see A.6.
[c] We repeatedly apply (4.16).

If we are only interested in the existence of a weak solution even weaker assumptions are sufficient:

Theorem 4.11. *Suppose D is convex. If μ and σ are bounded continuous on $[0,\infty)\times\overline{D}$, then on some probability space $(\Omega,\mathscr{A},\mathbb{P})$ there exists an r-dimensional Brownian motion B such that* (4.13) *has a solution.*

Proof. The proof can be found in [TANAKA 1979, Theorem 4.2]. □

4.2. SDEs with reflecting boundaries in regions with smooth boundaries

An early approach towards existence and uniqueness of solutions to stochastic differential equations with reflecting boundaries in regions with smooth boundaries is due to P. L. Lions and A. S. Sznitman [LIONS and SZNITMAN 1984]. They showed that D satisfying Condition B and C below as well as an admissibility condition suffices for the existence of a unique solution to the equation: deterministic Skorokhod problem. The admissibility condition means roughly that D can be approximated in some sense by smooth domains.
Y. Saisho later showed that we can drop this admissibility [SAISHO 1987]. Following the outline of [SAISHO 1987] we start by introducing the above mentioned conditions:

Condition B (uniform exterior sphere condition)**.** *There exists a constant $r>0$ such that*

$$\mathscr{N}_x=\mathscr{N}_{x,r}\neq\emptyset \text{ for any } x\in\partial D,$$

where $\mathscr{N}_x$ and $\mathscr{N}_{x,r}$ are as in (4.1)

Condition C. *There exist constants $\delta>0$ and $\beta\in[1,\infty)$ such that for any $x\in\partial D$ there exists a unit vector $\mathbf{l_x}$ satisfying*

$$\langle \mathbf{l_x},\mathbf{n}\rangle \geq 1/\beta \text{ for any } \mathbf{n}\in\bigcup_{y\in B_\delta(x)\cap\partial D}\mathscr{N}_y.$$

Before we start to explore how these conditions imply that a solution to the Skorokhod problem exists and is unique, we take a glance at Condition C and give the following counterexample.
We mentioned before that we are looking for a condition such that we can define a projection into $\overline{D}$. Condition B enables us to do that for points in $B_r(\overline{D})$, where r is the constant in Condition B. To see that, let $x\in\mathbb{R}^d$ be such that $\operatorname{dist}(x,\overline{D})<r$, then as $\overline{D}$ is closed there exists a point $\overline{x}\in\overline{D}$ such that $|\overline{x}-x|=\operatorname{dist}(x,\overline{D})$. This implies that we have $B_{|\overline{x}-x|}(x)\cap\overline{D}=\emptyset$ as otherwise we would find a point $\overline{x}'\in\overline{D}$ with

$$|\overline{x}'-x|<|\overline{x}-x|=\operatorname{dist}(x,\overline{D}).$$

Thus $\mathbf{n}:=(\overline{x}-x)/|\overline{x}-x|\in\mathscr{N}_{\overline{x}}$ and therefore $B_r(\overline{x}-r\mathbf{n})\cap\overline{D}=\emptyset$. As $B_{|\overline{x}-x|}(x)\subsetneq B_r(\overline{x}-r\mathbf{n})$, $\overline{x}$ is the unique point in $\overline{D}$ such that $|\overline{x}-x|=\operatorname{dist}(x,\overline{D})$. This means that the mapping

$$B_r(\overline{D})\to\overline{D},\, x\mapsto\overline{x}:=\operatorname{argmin}_{y\in\overline{D}}|x-y|$$

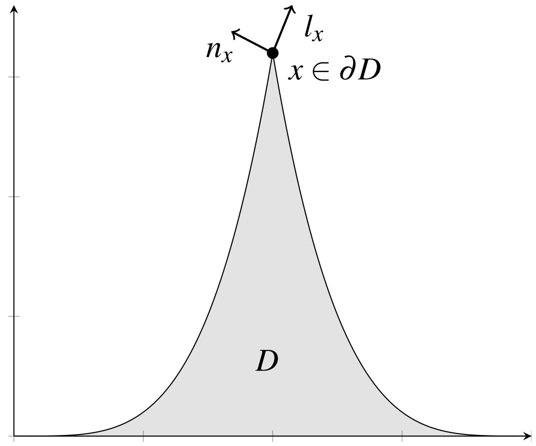

Figure 4.2.: Counterexample for Condition C: At the singularity x there exists, for any vector $\mathbf{l_x}$, an $\mathbf{n} \in \mathscr{N}$ such that $\langle \mathbf{l_x}, \mathbf{n} \rangle = 0$.

is well-defined.
For the next paragraph we assume that D satisfies Condition B. Analogously to Section 4.1 we turn towards jump functions first. But as we were only able to show that the projection is unique in $B_r(\overline{D})$, we have to restrict to functions with small jumps, i.e. we assume for $g \in D([0,\infty),\mathbb{R}^d)$ that $\sup_{t\in[0,\infty)} \Delta g(t) < r$, where r is as in Condition B and $\Delta g_n(t) := g_n(t) - g_n(t-)$ is the size of the jump occurring at $t \in [0,\infty)$. We put

$$f(t) := \begin{cases} g(t) + h(\tau_n) & \text{for} \quad \tau_n < t < \tau_{n+1} \\ \overline{g(t) + h(\tau_n)} & \text{for} \quad t = \tau_{n+1}, \end{cases}$$

where $\tau_n := \inf\{t > \tau_{n-1}; g(\tau_{n-1}) + h(t) \notin \overline{D}\}$ is exactly as in Lemma 4.2 and an analogous argumentation shows that h is associated to f.
Having established this theorem for step functions with small jumps we quote a lemma similar to Lemma 4.3:

Lemma 4.12. *Suppose D satisfies Condition B. Let* g, $\tilde{g} \in D([0,\infty),\mathbb{R}^d)$ *starting in* $g(0)$, $\tilde{g}(0) \in \overline{D}$, *and* f, $\tilde{f}$ *be solutions of* (4.2) *with* $f = g + h$ *and* $\tilde{f} = \tilde{g} + \tilde{h}$, *respectively. Then we have*

$$\begin{aligned} |f(t) - \tilde{f}(t)|^2 \le |g(t) - \tilde{g}(t)|^2 &+ \frac{1}{r}\int_0^t |f(s) - \tilde{f}(s)|^2 d(|h|(s) + |\tilde{h}|(s)) \\ &+ 2\int_0^t \langle g(t) - \tilde{g}(t) - g(s) + \tilde{g}(s), h(ds) - \tilde{h}(ds)\rangle, \end{aligned} \tag{4.18}$$

$$\begin{aligned} |f(t) - f(\tilde{t})|^2 \le |g(t) - g(\tilde{t})|^2 &+ \frac{1}{r}\int_0^t |f(s) - f(\tilde{t})|^2 d|h|(s) \\ &+ 2\int_{\tilde{t}}^t \langle g(t) - g(s), h(ds)\rangle \end{aligned} \tag{4.19}$$

for $0 \le \tilde{t} \le t$.

Proof. The proof can be found in [SAISHO 1987]. □

As done in Lemma 4.4 we want to use (4.18) to establish the uniqueness.
Considering two solutions f, $\tilde{f}$ of (4.2) we set $g = \tilde{g}$ in (4.18) and obtain directly

$$|f(t) - \tilde{f}(t)|^2 \le \frac{1}{r}\int_0^t |f(s) - \tilde{f}(s)|^2 \cdot \big(|h|(t) + |\tilde{h}|(t)\big) ds$$

and thus we can apply Gronwall's Lemma to obtain:

Lemma 4.13. *If D satisfies Condition B, then* (4.2) *has at most one solution.*

Now we want to get some impression on how to carry the result for step functions over to arbitrary $g \in C([0,\infty),\mathbb{R}^d)$. Note that as we have to ensure that g doesn't leave $B_r(\overline{D})$, we cannot take g out of $D([0,\infty),\mathbb{R}^d)$. In fact, for the following arguments we assume that g is uniformly continuous. We start by defining $g_n \in D([0,\infty),\mathbb{R}^d)$, for $n \in \mathbb{N}$, by

$$g_n(t) := g(k2^{-n}), \text{ if } k2^{-n} \leq t < (k+1)2^{-n}$$

and consider the Skorokhod equation $f_n = g_n + h_n$. With g uniformly continuous and $n \in \mathbb{N}$ chosen sufficiently large, g_n is a step function with small jumps $\sup_{t\in[0,\infty)} \Delta g_n(t) < r$. Thus, there exits a solution (f_n, h_n), this can be carried over to g.
Y. Saisho showed that the additional assumption that g is uniformly continuous is unnecessary.

Theorem 4.14. *Suppose D satisfies Condition B and C. Then for any $g \in C([0,\infty),\mathbb{R}^d)$ with $g(0) \in \overline{D}$ there exists a unique and continuous solution (f,h) of the equation* (4.2)*, and the solution depends continuously upon g.*

Proof. The proof can be found in [SAISHO 1987, Theorem 4.1]. □

As done in Lemma 4.7 we can find an estimate for the quadratic variation $|h|(s,t) := |h|(t) - |h|(s)$ in terms of $\Delta_{\tilde{t},t}(g) := \sup_{\tilde{t}\leq r<s\leq t} |g(r) - g(s)|$:

Theorem 4.15. *Suppose D satisfies Condition B and C. If (f,h) is the solution of* (4.2) *for $g \in C([0,\infty),\mathbb{R}^d)$, for any finite $T > 0$, we have*

$$|h|(\tilde{t},t) \leq K\Delta_{\tilde{t},t}(g),\ 0 \leq \tilde{t} < t \leq T,$$

where K is a constant depending only on the constants r, β, δ in Conditions B and C, T, $\|g\|_T$ and the modulus of uniform continuity of g on $[0,T]$.

Proof. The proof can be found in [SAISHO 1987, Theorem 4.2]. □

Next, we turn towards the stochastic differential equation, we restrict ourself to the case where the coefficients are not time dependent.

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)dW_t + d\Phi_t \\ X_0 &= x, \end{aligned} \tag{4.20}$$

P. L. Lions and A. S. Sznitman showed in [LIONS and SZNITMAN 1984] that assuming the admissibility condition and

Condition D. *There exists a function f in $C^2(R,R^d)$ which is bounded together with its first and second partial derivatives such that $\exists\gamma > 0$, $\forall x \in \partial D$, $\forall y \in \overline{D}$, $\forall \mathbf{n} \in \mathscr{N}_x$*

$$\langle y - x, \mathbf{n}\rangle + \frac{1}{\gamma}\langle \nabla f(x), \mathbf{n}\rangle |y - x|^2 \geq 0. \tag{4.21}$$

there exists a unique solution to (4.20) for Lipschitz-continuous coefficients μ and σ. This result was later on improved by Y. Saisho, who showed that assuming only Condition B and C one can obtain a local version of Condition D and that the admissibility condition is unnecessary:

Theorem 4.16. *Suppose D satisfies Conditions B and C. If the coefficients μ and σ are bounded and Lipschitz-continuous then there exists a unique strong solution of* (4.20).

Proof. We sketch the proof given in [SAISHO 1987, Theorem 5.1] and start by considering the discretized stochastic differential equation with reflection for $n \in \mathbb{N}$

$$\begin{aligned} dX_t^{(n)} &= \mu(X_{h_n(t)}^{(n)})dt + \sigma(X_{h_n(t)}^{(n)})dW_t + \Phi_t^{(n)}, \\ X_0^{(n)} &= X_0, \end{aligned} \tag{4.22}$$

where

$$h_n(0) = 0, \text{ and } h_n(t) = (k-1)\,2^{-n} \text{ for } (k-1)\,2^{-n} \le t \le k\,2^{-n} \text{ for } k \ge 1.$$

We can apply Theorem 4.14 to obtain the unique solution of (4.22) as once $X_t^{(n)}$ is obtained for $0 \le t \le k2^{-n}$, $X_t^{(n)}$ is uniquely determined on $(k2^{-n},(k+1)\,2^{-n}]$ as the solution to the deterministic Skorokhod problem

$$X_t^{(n)} = X_{k2^{-n}}^{(n)} + \mu(X_{k2^{-n}}^{(n)})\left(t - k2^{-n}\right) + \sigma(X_{k2^{-n}}^{(n)})\,(W_t - W_{k2^{-n}}) + \Phi_t^{(n)}.$$

We put

$$Y_t^{(n)} := X_0 + \int_0^t \mu(X_{h_n(s)}^{(n)})ds + \int_0^t \sigma(X_{h_n(s)}^{(n)})dW_s, \tag{4.23}$$

and denote by $\mathbb{P}_n$ the probability measure on $C([0,T],\mathbb{R}^d \times \mathbb{R}^d)$ associated to the process $(B_t, Y_t^{(n)})_{0 \le t \le T}$, for an arbitrary $T > 0$. It can be shown that $(\mathbb{P}_n)_{n \in \mathbb{N}}$ is tight (see A.9 for a definition and [SAISHO 1987, Lemma 5.1] for the derivation).
Thus, by Prohorov's Theorem (see A.10) there exists a weakly convergent subsequence, which for simplicity we denote as well by $(\mathbb{P}_n)_{n \in \mathbb{N}}$. This enables us to apply Skorokhod's Representation Theorem (see A.11) to obtain on a suitable probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ a sequence of processes

$$\left((\tilde{B}_t^{(n)}, \tilde{Y}_t^{(n)})_{0 \le t \le T}\right)_{n \in \mathbb{N}}, \tag{4.24}$$

such that $(\tilde{B}_t^{(n)}, \tilde{Y}_t^{(n)})_{0 \le t \le T}$ is identical in law to $(B_t^{(n)}, Y_t^{(n)})_{0 \le t \le T}$, for each $n \in \mathbb{N}$, converging almost surely uniformly on $[0,T]$ as $n \to \infty$ to some process $(\tilde{B}_t^{(n)}, \tilde{Y}_t^{(n)})$.
Let $(\tilde{X}_t^{(n)}, \tilde{\Phi}_t^{(n)})_{0 \le t \le T}$ and $(\tilde{X}_t, \tilde{\Phi}_t)_{0 \le t \le T}$ be the solution to the Skorokhod equations

$$\tilde{X}_t^{(n)} = \tilde{Y}_t^{(n)} + \tilde{\Phi}_t^{(n)}, \tag{4.25}$$

$$\tilde{X}_t = \tilde{Y}_t + \tilde{\Phi}_t, \tag{4.26}$$

respectively. The continuity result in Theorem 4.14 implies that $(\tilde{X}_t^{(n)}, \tilde{\Phi}_t^{(n)})_{0 \le t \le T}$ converges almost surely uniformly on $[0,T]$ as $n \to \infty$ to $(\tilde{X}_t, \tilde{\Phi}_t)_{0 \le t \le T}$. This can be used to show that $(\tilde{X}_t, \tilde{\Phi}_t)$ is a solution of the Skorokhod SDE

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t \mu(\tilde{X}_s)ds + \int_0^t \sigma(\tilde{X}_s)dW_s + \tilde{\Phi}_t. \tag{4.27}$$

We just roughly sketch how a local version of Condition D is established and used to prove the uniqueness of the solution. We denote the constants in Condition B and C by r, β and δ, and notice that for δ and $\gamma := {}^{2r}/_{\beta}$ and any $x_0 \in \partial D$, we can define $f(x) := \langle \mathbf{l}, x\rangle$ for the unit vector given by Condition C. This implies directly for any $x \in B_\delta(x_0) \cap \partial D$ and $\mathbf{n} \in \mathscr{N}_x$

$$\frac{1}{\gamma}\langle \nabla f(x), \mathbf{n}\rangle = \frac{1}{\gamma}\langle \mathbf{l}, \mathbf{n}\rangle \geq \frac{1}{2r}.$$

Now it suffices to note that $\mathbf{n} \in \mathscr{N}_{x,r}$ implies that

$$r^2 \geq |y - x - r\mathbf{n}|^2 = |y-x|^2 + 2r\langle y-x, \mathbf{n}\rangle + r^2,$$

from which we obtain $\langle y-x, \mathbf{n}\rangle + \frac{1}{2r}|y-x|^2 \geq 0$. Put together we see that f satisfies

$$\langle y-x, \mathbf{n}\rangle + \frac{1}{\gamma}\langle \nabla f(x), \mathbf{n}\rangle |y-x|^2 \geq 0. \tag{4.21}$$

for any $x \in B_\delta(x_0) \cap \partial D$, $y \in B_\delta(x_0) \cap \overline{D}$ and $\mathbf{n} \in \mathscr{N}_x$.
This can be used to show that assuming the support of μ and σ is included in $B_\delta(x_0)$ for some $x_0 \in \partial D$ two solutions are identical almost surely. A more detailed explanation and survey on how to carry this over to general μ and σ can be found in [SAISHO 1987, Theorem 5.1]). □

5. Multi-dimensional diffusions with oblique reflection

So far we have only considered reflection along the normal vector on the domain D. We can generalize this by exchanging the normal vector in the definition of the Skorokhod problem by an arbitrary vector. The theory concerning this oblique reflection is far less developed and in particular the question of uniqueness is still open.
We start by taking a look at the theoretical results developed by C. Costantini published in [COSTANTINI 1992]. As including all the proofs goes beyond the scope of this thesis, we omit most of them. Instead we consider in Section 5.2 the example of the positive orthant with constant direction of reflection on each limiting hyperplane covering all the necessary proofs.

5.1. Theoretical Results

We consider a domain D in $\mathbb{R}^d$, and a multivalued vector field Γ such that Γ_x is a closed convex cone in $\mathbb{R}^d$ for every $x \in \partial D$. As we mostly need reflection vectors with norm 1, we define $\Gamma_x^1 := \Gamma_x \cap S_1(0)$ for $x \in \partial D$. The deterministic Skorokhod problem then reads:
Given a function $g \in D([0,\infty),\mathbb{R}^d)$ starting in $g(0) \in \overline{D}$, find a solution (f,h) of

$$f = g + h,$$

where f is in $D([0,\infty),\overline{D})$ and h satisfies

1. h is of bounded variation.

2. The set $\{t \in [0,\infty); f(t) \in D\}$ has $d|h|$-measure zero.

3. In the representation
$$h(t) = \int_0^t \gamma(s) d|h|(s)$$
the vector $\gamma(s)$ is in $\Gamma_{f(s)}$ and $|\gamma(s)|$ for almost all s with respect to the measure $d|h|$.

Again, the total variation $|h|$ is called the local time of the solution.
Throughout this section we will follow the thorough survey concerning oblique reflection by C. Costantini [COSTANTINI 1992]. His results are considered to be the most general concerning the existence of solutions to the Skorokhod problem. To get some impression on the earlier work done concerning this topic we start by mentioning the results obtained by P. L. Lions and A. S. Sznitman [LIONS and SZNITMAN 1984].
They considered a bounded and smooth domain D and showed, assuming that Γ is twice continuous differentiable and the angle between $n \in \mathscr{N}_x$ and $\gamma \in \Gamma_x$ for $x \in \partial D$ is uniformly bounded away from $\frac{\pi}{2}$, the existence of a solution (f,h) to the deterministic Skorokhod problem. Additionally, they gave a counterexample for the uniqueness of the solution and showed that if either

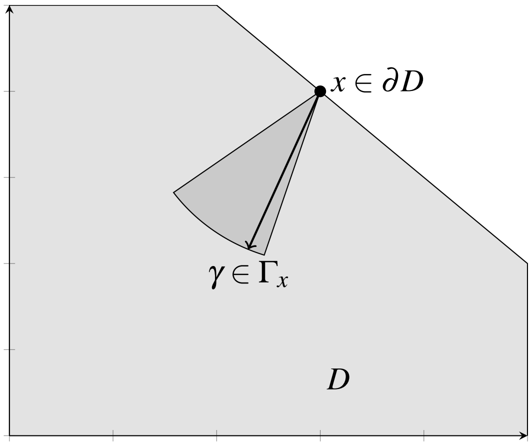

Figure 5.1.: The vector γ can be chosen arbitrarily from the cone Γ_x.

g is of bounded variation or if $\mathscr{N}$ can be linearly transformed into Γ, i.e. $\exists A \in \mathrm{Mat}(d \times d, \mathbb{R})$ such that $A\gamma = n$ for $\gamma \in \Gamma_x$ and $n \in \mathscr{N}_x$ for any $x \in \partial D$, the solution is unique. Lastly, they obtained under the same assumptions that the stochastic differential equation (5.16) with Lipschitz continuous coefficients has a unique solution. For details and proofs we refer to [Lions and Sznitman 1984].

Our approach is similar to the one applied in Chapter 4: we try to find conditions such that we can define a projection into $\overline{D}$. For this setting this is much more demanding and we have to introduce some variables measuring properties of Γ. For $x \in \partial D$ and $\rho > 0$ define:

$$a(x,\rho) := \max_{v \in S_1(0)} \min_{y \in \partial D \cap \overline{B_\rho(x)}} \min_{\gamma \in \Gamma_y^1} \langle \gamma, v \rangle, \tag{5.1}$$

$$c(x) := \sup_{z \in \overline{D}, z \neq x} \max_{\gamma \in \Gamma_x^1} (\langle \gamma, \frac{x-z}{|x-z|} \rangle \vee 0), \tag{5.2}$$

$$c(x,\rho) := \max_{y \in \partial D \cap \overline{B_\rho(x)}} c(y), \tag{5.3}$$

$$\tilde{c}(x,\rho) := \max_{y \in \partial D \cap \overline{B_\rho(x)}} \sup_{z \in \overline{D} \cap \overline{B_\rho(x)} z \neq y} \max_{\gamma \in \Gamma_y^1} (\langle \gamma, \frac{y-z}{|y-z|} \rangle \vee 0), \tag{5.4}$$

$$e(x,\rho) := \frac{c(x,\rho)}{a^2(x,\rho) \vee a(x,\rho)/2}, \tag{5.5}$$

$$\tilde{e}(x,\rho) := \frac{\tilde{c}(x,\rho)}{a^2(x,\rho) \vee a(x,\rho)/2}. \tag{5.6}$$

$a(x,\rho)$ can be understood as a measure how much Γ varies around $x \in \partial D$, with values around 0 standing vectors distributed all over a half-space. As we want little variation we will later on formulate a condition ensuring that a is bounded away from 0. While $c(x,\rho)$ and $\tilde{c}(x,\rho)$ can be understood as a measure how skewed Γ is and how concave ∂D is.

We will not go into any detail about the proofs, still we want to state some of the important intermediate results and the involved conditions on $a(x,\rho)$ and $e(x,\rho)$.

Theorem 5.1. *If*

$$\lim_{\rho\to 0} \inf_{x\in\partial D} a(x,\rho) = a > 0, \tag{5.7}$$

$$\lim_{\rho\to 0} \sup_{x\in\partial D} e(x,\rho) = e < 1, \tag{5.8}$$

then, for every $g \in D([0,\infty),\mathbb{R}^d)$ with $g(0) \in \overline{D}$ and for any $T > 0$, there exist positive constants $K(g,T)$ and $\tilde{K}(g,T)$ such that, for any solution (f,h) to the Skorokhod problem for (D,Γ,g) we have the following estimates:

$$\begin{aligned} \Delta_{\tilde{t},t}(f) &\leq K(g,T)\Delta_{s,t}(g), \quad 0 \leq \tilde{t} \leq t \leq T, \\ |h|_{\tilde{t},t} &\leq \tilde{K}(g,T)\Delta_{s,t}(g), \quad 0 \leq \tilde{t} \leq t \leq T. \end{aligned} \tag{5.9}$$

For a $G \subset \{g \in D([0,\infty),\mathbb{R}^d);\, g(0) \in \overline{D}\}$ relatively compact in the Skorokhod topology K and $\tilde{K}$ are bounded, i.e.

$$\sup_{g\in G} K(g,T) = K_T < \infty, \qquad \sup_{g\in G} \tilde{K}(g,T) = \tilde{K}_T < \infty. \tag{5.10}$$

Proof. The proof can be found in [COSTANTINI 1992, Theorem 2.2]. □

The conditions (5.7) and (5.8) are fairly restrictive, we can weaken them slightly at the cost of an additional condition on the jump size of the solution.

Theorem 5.2. *If there exists a constant $\rho_0 > 0$ such that*

$$\inf_{x\in\partial D} a(x,\rho_0) = a < 0, \tag{5.11}$$

$$\sup_{x\in\partial D} \tilde{e}(x,\rho_0) = \tilde{e} < 1, \tag{5.12}$$

then for any $g \in D([0,\infty),\mathbb{R}^d)$ with $g(0) \in \overline{D}$, for any $T > 0$, (5.9) and (5.10) hold for any solution (f,h) to the Skorokhod problem for (D,Γ,g) such that $x \in \overline{D^{\rho_0}([0,\infty),\overline{D})}$.

Proof. The proof can be found in [COSTANTINI 1992, Theorem 2.4]. □

From (5.9) we deduce directly that the continuity of g implies the continuity of f. Using Theorem 5.1 and Theorem 5.2 we can connect this with the conditions (5.7) and (5.8) respectively (5.11) and (5.12).

Corollary 5.3. *If (5.7) and (5.8) hold, then for every $g \in C([0,\infty),\mathbb{R}^d)$ with $g(0) \in \overline{D}$ any solution (f,h) to the Skorokhod problem for (D,Γ,g) is continuous. If (5.11) and (5.12) hold, then for every $g \in C([0,\infty),\mathbb{R}^d)$ with $g(0) \in \overline{D}$ any solution (f,h) to the Skorokhod problem for (D,Γ,g) having small jumps $\sup_{t\leq T} \Delta f(t) < \rho_0$ with ρ_0 as in (5.11) and (5.12), is continuous.*

Assuming the convexity of D it can be shown that the conditions in Theorem 5.1 and 5.2 are satisfied under various additional assumptions, see [COSTANTINI 1992, Proposition 2.3 and 2.5].

We see that Theorem 5.1 and 5.2 yield that the Skorokhod map $g \mapsto f$ preserves relative compactness. C. Costantini obtained a stronger result, allowing us to approximate a function g by $(g_n)_{n\in\mathbb{N}}$ and obtain a solution to (D,Γ,g) by taking a limit point of the sequence of solutions $(f_n)_{n\in\mathbb{N}}$ of (D,Γ,g_n).

Theorem 5.4. *Let (f_n,h_n) be a solution to the Skorokhod problem (D,Γ,g_n), $g_n \in D([0,\infty),\mathbb{R}^d)$ such that $g_n(0) \in \overline{D}$ for all $n \in \mathbb{N}$ and $(g_n)_{n\in\mathbb{N}}$ converges to a function g. Then any limit point of $(f_n)_{n\in\mathbb{N}}$ in $D([0,\infty),\overline{D}\times\mathbb{R}^d)$ is a solution to the Skorokhod problem for (D,Γ,g) if either of the following is fulfilled:*

1. (5.7) *and* (5.8) *hold.*
2. (5.11) *and* (5.12) *hold as well as the jump size satisfies* $\sup_{t\le T}\Delta f_n(t) < \rho_0$ *for almost all* $n \in \mathbb{N}$.

Proof. The proof can be found in [COSTANTINI 1992, Theorem 3.1 and 3.2]. □

Now we are equipped to turn towards the existence of a projection into $\overline{D}$ and give some sufficient condition for existence and uniqueness. For the projection $x \mapsto \bar{x}$ we want it to be along Γ, i.e. for $x \in \mathbb{R}^d \setminus \overline{D}$ we want $\bar{x}-x$ to be in $\Gamma_{\bar{x}}$. For normal reflection we have seen previously that the uniform exterior sphere condition is sufficient for the existence and uniqueness of the projection locally around $\overline{D}$. In the following we will need this condition again and additionally

Condition E. *The set $G^\Gamma := \{(x,\gamma); x\in\partial D, \gamma\in\Gamma_x\}$ is closed.*

Theorem 5.5. *Suppose D satisfies the uniform exterior sphere condition, as well as D and Γ satisfy Condition E. If there exists a continuous map $Q : G^{\mathscr{N}} \to \mathbb{R}^d$ such that*

$$\begin{aligned} Q(x,\mathscr{N}_x) &= \Gamma_x, && x\in\partial D,\\ Q(x,\lambda\mathbf{n}) &= \lambda Q(x,\mathbf{n}) && \forall\lambda\ge 0, x\in\partial D, \forall\mathbf{n}\in\mathscr{N}_x, \end{aligned} \tag{5.13}$$

$$\begin{aligned} \sup_{x\in\partial D}\max_{\mathbf{n}\,\in\,\mathscr{N}_x^1} |Q(x,\mathbf{n})| &:= \|Q\| < \infty,\\ \inf_{x\,\in\,\partial D}\min_{\mathbf{n}\in\mathscr{N}_x^1} \langle \mathbf{n}, Q(x,\mathbf{n})\rangle &> 0. \end{aligned} \tag{5.14}$$

Then there exists $\delta_0 > 0$ such that every $x \in \mathbb{R}^d \setminus D$ with $d(x,\overline{D}) < \delta_0$ admits at least one projection $\bar{x}$ on ∂D along Γ satisfying

$$|\bar{x}-x| \le \frac{q_G/\|Q\|_G}{1-\sqrt{1-q_G/\|Q\|_G{}^2}}\, d(\bar{x},\overline{D}). \tag{5.15}$$

If D is convex then the same assertion holds for every $x\in\mathbb{R}^d\setminus D$, without any restriction on $d(x,\overline{D})$.

Proof. The proof can be found in [COSTANTINI 1992, Theorem 4.1]. □

In [COSTANTINI 1992] there is more theory concerning the existence of such maps $Q : G^{\mathscr{N}} \to \mathbb{R}^d$ and an alternative to Theorem 5.5. We omit this and come directly to the theorems concerning the existence of solutions to the Skorokhod problem.

Theorem 5.6. *Suppose D satisfies the uniform exterior sphere condition, as well as D and Γ satisfy Condition E. Let $Q : G^{\mathscr{N}} \to \mathbb{R}^d$ be continuous and satisfy* (5.13) *and* (5.14).

1. *If* (5.7) *and* (5.8) *or* (5.11) *and* (5.12) *hold, then for each $g \in C([0,\infty),\mathbb{R}^d)$ starting in $g(0)\in\overline{D}$ there exists a solution $(f,h)\in C([0,\infty),\overline{D}\times\mathbb{R}^d)$ to the Skorokhod problem for (D,Γ,g).*

2. *If* (5.7) *and* (5.8) *hold, then for each* $g \in D([0,\infty),\mathbb{R}^d)$ *starting in* $g(0) \in \overline{D}$ *with jump size* $\sup_{t\geq 0} \Delta g(t) < \delta_0$ *there exists a solution* $(f,h) \in D([0,\infty),\overline{D}\times\mathbb{R}^d)$ *to the Skorokhod problem for* (D,Γ,g).

3. *If* (5.11) *and* (5.12) *hold, then for each* $g \in D([0,\infty),\mathbb{R}^d)$ *starting in* $g(0) \in \overline{D}$ *with jump size* $\sup_{t\geq 0} \Delta g(t) < \delta_0 \vee \overline{\rho}_0$ *there exists a solution* $(f,h) \in D^{\rho_0}([0,\infty),\overline{D}\times\mathbb{R}^d)$ *with jump size* $\sup_{t\geq 0} \Delta f(t) < \rho_0$ *to the Skorokhod problem for* (D,Γ,g)*, where*

$$\overline{\rho}_0 = \rho_0 \cdot \left(1 - \sqrt{1 - \frac{q_G}{\|Q\|_G}^2}\right) \cdot \frac{\|Q\|_G}{q_G}.$$

Proof. The proof can be found in [Costantini 1992, Theorem 5.2]. □

From this we can derive immediately the stochastic version:

Theorem 5.7. *Suppose D satisfies the uniform exterior sphere condition and D and Γ satisfy Condition E. Let $Q : G^{\mathscr{N}} \to \mathbb{R}^d$ be continuous and satisfy* (5.13) *and* (5.14)*. If* (5.7) *and* (5.8) *or* (5.11) *and* (5.12) *hold, then:*

1. *For each standard Brownian motion W on a probability space $(\Omega,\mathscr{F},P)$ there exists a continuous stochastic process (X,ϕ) on $(\Omega,\mathscr{F},P)$ such that (X,ϕ) is almost surely the solution to the Skorokhod problem for (D,Γ,W).*

2. *There exists a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{P})$ and continuous processes $\tilde{W}$ and (X,ϕ), both defined on $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{P})$, such that $\tilde{W}$ is a standard Brownian motion and $(\tilde{X},\tilde{\phi})$ is a solution to the Skorokhod problem for $(D,\Gamma,\tilde{W})$.*

Proof. The proof can be found in [Costantini 1992, Theorem 5.3]. □

Before stating the equivalent for SDE, we have to define the weak solution of an SDE.

Definition 5.8. *A triple $((\Omega,\mathscr{A},P,\mathscr{F}),W,X)$, where $(\Omega,\mathscr{A},P,\mathscr{F})$ is a filtered probability space, W and X are $\mathbb{R}^d$-valued stochastic processes on $(\Omega,\mathscr{A},P)$, is a* weak solution *to the stochastic differential equation with coefficients μ and σ in $\overline{D}$ with reflection along Γ on ∂D, with initial distribution P_0, if W is an $\mathscr{F}$-adapted standard Brownian motion, X is $\mathscr{F}$-adapted, the law of X_0 is P_0, and there exists an $\mathscr{F}$-adapted stochastic process ϕ of bounded variation, such that (X,ϕ) is almost surely a solution to the Skorokhod problem for $(D,\Gamma,\int_0^\cdot b(X_s)ds + \int_0^\cdot \sigma(X_s)dW_s)$, i.e. (X,ϕ) satisfies, almost surely,*

$$\begin{aligned} X_t &= X_0 + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s + \phi_t, \text{ for } t \in [0,\infty) \\ \phi_t &= \int_0^t \gamma_s\, d|\phi|_s, \text{ where } \gamma_s \in \Gamma^1_{X_s}\, d|\phi| - a.e., \text{ for } t \in [0,\infty) \\ d|\phi|(\{t \in [0,\infty); X_t &\in D\}) = 0, \end{aligned} \tag{5.16}$$

Theorem 5.9. *Suppose D satisfies the uniform exterior sphere condition and D and Γ satisfy Condition E. If there exists a continuous map $Q : G^{\mathscr{N}} \to \mathbb{R}^d$ satisfying* (5.13) *and* (5.14)*, and either* (5.7) *and* (5.8) *or* (5.11) *and* (5.12) *hold, then there exists a weak solution to the stochastic differential equation with reflection* (5.16)*.*

Proof. The proof can be found in [Costantini 1992, Theorem 5.4]. □

5.2. Reflected Brownian motion on an Orthant

After having established the theory we consider next the example of the nonnegative orthant $D := \{x \in \mathbb{R}^d; x_k > 0 \text{ for all } k \in \{1,...,d\}\}$ with constant direction of reflection on each of the boundaries $\{x \in \overline{D}; x_l = 0\}$ solved in [HARRISON and REIMAN 1981]. For this specific setting, we can easily give a proof for existence and uniqueness of the solution of the Skorokhod problem. Before continuing with the deterministic case, we start by setting the direction of reflection on $\{x \in \overline{D}; x_l = 0\}$ to be the lth row of the reflection matrix $I-Q$, where Q is a nonnegative $d \times d$-matrix with zeros on the diagonal and spectral radius strictly smaller than one. As the direction of reflection has to have a positive inward normal the ones on the diagonal of $I-Q$ are just a normalization while the non-negativity of Q means that we restrict to the case where reflection is downward (towards the origin). The restriction on the spectral radius is essential as J. M. Harrison and M. I. Reiman showed that without it there doesn't exist a solution for the stochastic Skorokhod problem.

Theorem 5.10. *For $g \in C([0,\infty),\mathbb{R}^d)$ starting in $g(0) \in \overline{D}$ there exists a unique solution $(f,h) \in C([0,\infty),\overline{D}) \times C([0,\infty),\overline{D})$ satisfying*

$$f = g + (I-Q)h \tag{5.17}$$

$$h \text{ is nondecreasing with } h(0) = 0 \tag{5.18}$$

and

$$h_l \text{ increases only at those times } t \text{ where } f_l(t) = 0 \tag{5.19}$$

for all $l \in \{1,...,d\}$.
Moreover, the solution (f,h) depends continuously on g and the restrictions of f and h to $[0,T]$ depend only on the restriction of g to $[0,T]$. We define $\tilde{g}(t) = g(T+t) + f(T) - g(T)$, $\tilde{h}(t) = h(T+t) - h(T)$ and $\tilde{f}(t) = f(T+t)$, then

$$(\tilde{f},\tilde{h}) \text{ is the unique solution to the corresponding problem for } \tilde{g}. \tag{5.20}$$

Proof. Throughout the proof we will use the maximum absolute row sum norm $\|\cdot\|_\infty$. As Q is nonnegative and has spectral radius smaller than one, there exists a diagonal matrix Γ, having positive diagonal elements, such that the nonnegative matrix $Q^* = \Gamma^{-1}Q\Gamma$ satisfies $\|Q^*\|_\infty < 1$. Having found a solution (f^*,h^*) corresponding to (D,Q^*,g^*) we can transform this to a solution $(f^*\Gamma^{-1},h^*\Gamma^{-1})$ corresponding to $(D,Q,x^*\Gamma^{-1})$. Thus, we can assume without loss of generality that $\|Q\|_\infty = \alpha < 1$.
We define $C := \{f \in C([0,\infty),\mathbb{R}^d); f \text{ is nondecreasing and } f(0) = 0\}$ and set $\pi : C \to C$ such that

$$\pi(h)(t) = \sup_{s\le t}(Qh(s) - g(s))^+, \tag{5.21}$$

for $t \ge 0$ (positive part and supremum are to be computed component-wise).
Given (5.17) and (5.18) the conditions $f \in C([0,\infty),\overline{D})$ and (5.19) are equivalent to

$$h = \pi(h) \tag{5.22}$$

with (5.21) it is easy to see that (5.22) implies $f \in C([0,\infty),\overline{D})$ and (5.19). Thus, it remains to show that $f \in C([0,\infty),\overline{D})$ and (5.19) imply (5.22). Suppose $f \in C([0,\infty),\overline{D})$ and (5.19), then component-wise

$$\pi(h)(t) = \sup_{s\le t}\big(Qh(s) - g(s)\big)^+ \overset{(5.17)}{=} \sup_{s\le t}\big(h(s) - f(s)\big)^+ \le \sup_{s\le t} h(s) \overset{(5.18)}{\le} h(t).$$

Assume $\pi_l(h)(t) > h_l(t)$ for some $l \in \{1,...,d\}$ and $t \geq 0$, then $\pi_l(h)(\tilde{t}) > h_l(\tilde{t})$ for some $\tilde{t} \geq 0$ that is a point of increase for h_l and

$$f_l(\tilde{t}) \overset{(5.17)}{=} g_l(\tilde{t}) + (I-Q)_l h(\tilde{t}) \geq h_l(\tilde{t}) + \inf_{0 \leq s \leq \tilde{t}} \big(g_l(s) - Qh_l(s)\big) \overset{(5.21)}{=} h_l(\tilde{t}) - \pi(h)(\tilde{t}) > 0.$$

This contradicts that $\tilde{t}$ is a point of increase for h_l, i.e., $h = \pi(h)$.
We define the norm $\|h\| := \max_{1 \leq j \leq d} \sup_{t \leq T} |h_j(t)|$ for $h \in C([0,T],\mathbb{R}^d)$ and observe that the normed space $(C, \|\cdot\|)$ is complete.
Next, we show that π as a map $C \to C$ is a contraction

$$\begin{aligned}
\|\pi(h) - \pi(\tilde{h})\| &= \|\sup_{s \leq \cdot} \Big(Qh(s) - g(s)\Big)^+ - \sup_{s \leq \cdot} \Big(Q\tilde{h}(s) - g(s)\Big)^+ \| \\
&\leq \|\sup_{s \leq \cdot} \Big(Q(h(s) - \tilde{h}(s))\Big)^+ \| \\
&= \|Q(h - \tilde{h})\| \\
&\leq \|Q\|_\infty \cdot \|h - \tilde{h}\| \\
&= \alpha \|h - \tilde{h}\|,
\end{aligned}$$

for $h, \tilde{h} \in C$.
Thus, we have a unique fixed point $h = \lim_{n \to \infty} \in C$, that can be constructed by the following recursive definition

$$\begin{aligned}
h^{(0)} &= 0 \\
h^{(n+1)} &= \pi(h^{(n)})
\end{aligned} \tag{5.23}$$

for $n \in \mathbb{N}$.
As our choice of $T > 0$ was arbitrary, we have the uniform convergence of $(h^{(n)})_{n \in \mathbb{N}}$ to h on compact intervals $[0,T]$. With this we can define f in terms of g and h by (5.17).
From the construction we obtain directly that the restrictions of f and h to $[0,T]$ depend only on the restriction of g to $[0,T]$.
The only thing left is to prove the continuity property, for g, $\tilde{g} \in C([0,T],\mathbb{R}^d)$ starting in $g(0)$ and $\tilde{g}(0) \in \overline{D}$, respectively, we consider the solution $h(g)$ and $h(\tilde{g})$ constructed as above.

$$\begin{aligned}
\|h^{(n+1)}(g) - h^{(n+1)}(\tilde{g})\| &= \|\sup_{s \leq \cdot} \Big(Qh^{(n)}(g)(s) - g(s)\Big)^+ - \sup_{s \leq \cdot} \Big(Qh^{(n)}(\tilde{g})(s) - \tilde{g}(s)\Big)^+ \| \\
&\leq \|g - \tilde{g}\| + \|\sup_{s \leq \cdot} Q(h^{(n)}(g)(s) - h^{(n)}(\tilde{g})(s))\| \\
&\leq \|g - \tilde{g}\| + \alpha \|h^{(n)}(g) - h^{(n)}(\tilde{g})\|,
\end{aligned}$$

as $\|h^{(0)}(g) - h^{(0)}(\tilde{g})\| = 0$ we have $\|h(g) - h(\tilde{g})\| \leq \|g-\tilde{g}\|/(1-\alpha)$. Thus, $g \mapsto h(g)$ is continuously in the topology of uniform convergence on compact intervals, making the continuity of $g \mapsto f(g)$ obvious. □

We can use this results for the deterministic case directly to approach the stochastic case. To do so we consider the probability space $(\Omega, \mathscr{A}, P)$ together with the d-dimensional Brownian motion $(W_t)_{t \geq 0}$ with covariance matrix σ and drift μ starting in $W_0 \in \overline{D}$ and set $\mathscr{F}_t := \sigma(W_s; 0 \leq s \leq t)$.

A solution (X, ϕ) to the stochastic Skorokhod problem $(\overline{D}, I-Q, W)$ satisfies

$$X = W + (I-Q)\phi, \tag{5.24}$$
$$X_t \in \overline{D} \text{ almost surely for all } t \in [0,\infty), \tag{5.25}$$
$$\phi \text{ is almost surely continuous and nondecreasing starting in } \phi_0 = 0, \tag{5.26}$$
$$\phi_l \text{ increases almost surely only at those times } t \text{ where } X_l(t) = 0. \tag{5.27}$$

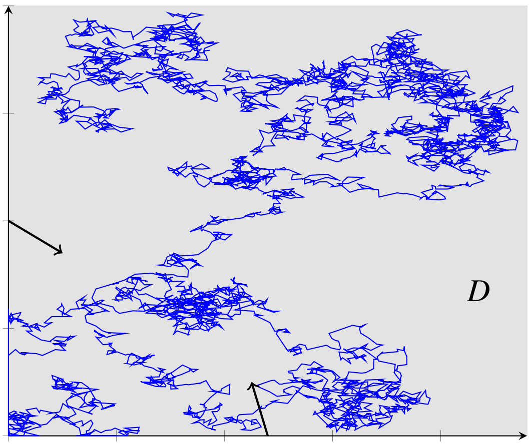

Figure 5.2.: The picture shows a reflected standard Brownian motion —— on the positive orthant and the direction of reflection ⟶ on each surface.

Corollary 5.11. *We set the processes X and ϕ to be the image of W under $g \mapsto f(g)$ and $g \mapsto h(g)$, respectively. Then,*

1. *The above defined processes X and ϕ are measurable with respect to $\mathscr{F}$*
2. *The processes X and ϕ uniquely satisfy* (5.24)-(5.27).
3. *X is a Markov process with stationary transition probabilities.*

Proof. As the maps $g \mapsto f(g)$ and $g \mapsto h(g)$ are continuous and the restriction of f and h to $[0,T]$ depend only on the restriction of g to $[0,T]$, we obtain 1. from the measurability of W. 2. is a direct consequence of the corresponding results for the deterministic case and our construction of X and ϕ and 3. follows from (5.20). □

We finish with decomposing W into $W^{(0)} + \mu t$, where $W^{(0)}$ is a Brownian motion with zero drift and covariance matrix σ, and conclude that

$$X_t = \underbrace{W_t^{(0)}}_{\text{martingale}} + \underbrace{\mu t + (I-Q)\phi_t}_{\text{of finite variation}} \tag{5.28}$$

is a semimartingale.

6. The Skorokhod problem with time-dependent reflecting boundaries

The last generalization for the Skorokhod problem defined in Chapter 3, we are going to consider, is the Skorokhod problem on time-dependent domains. We restrict ourselves to the one-dimensional case of time-dependent intervals described in [BURDZY et al. 2009]. For a review concerning the notationally much more demanding multi-dimensional case see e.g. [NYSTRÖM and ÖNSKOG 2010].
To define the Skorokhod problem on time-dependent intervals we introduce $D^+([0,\infty),\mathbb{R})$ and $D^-([0,\infty),\mathbb{R})$ for the set of càdlàg functions on $[0,\infty)$ with values in $(-\infty,\infty]$ and $[-\infty,\infty)$, respectively, to allow for infinite endpoint of the interval:

Definition 6.1 (Skorokhod problem on $[l,r]$)**.** *For $l \in D^-([0,\infty),\mathbb{R})$ and $r \in D^+([0,\infty),\mathbb{R})$, such that $l \leq r$, $(f,h) \in D([0,\infty),\mathbb{R}) \times D([0,\infty),\mathbb{R})$ is said to solve the Skorokhod problem for given $g \in D([0,\infty),\mathbb{R})$ if and only if it satisfies the following properties:*

1. *$f(t) = g(t) + h(t) \in [l(t), r(t)]$ for every $0 \leq t \leq \infty$.*

2. *We can decompose $h = h_l - h_r$ into two non-decreasing functions h_l and h_r such that*

$$\begin{aligned} &\text{The set } \{t \in [0,\infty);\, f(t) > l(t)\} \text{ has } dh_l\text{-measure zero,} \\ &\text{and } \{t \in [0,\infty);\, f(t) < r(t)\} \text{ has } dh_r\text{-measure zero.} \end{aligned} \tag{6.1}$$

If (f,h) is the unique solution to the Skorokhod problem on $[l,r]$ for g then we write $f = \Gamma_{l,r}(g)$, and refer to $\Gamma_{l,r}$ as the associated Skorokhod map. The pair (h_l,h_r) will be referred to as the constraining process associated with the Skorokhod problem.

We can slightly extend this by dropping the assumptions that the constraining process (h_l,h_r) are of bounded variation and define the extended Skorokhod problem. We will see that under mild conditions both problems are actually equivalent.

Definition 6.2 (Extended Skorokhod problem on $[l,r]$)**.** *For $l \in D^-([0,\infty),\mathbb{R})$ and $r \in D^+([0,\infty),\mathbb{R})$, such that $l \leq r$, $(f,h) \in D([0,\infty),\mathbb{R}) \times D([0,\infty),\mathbb{R})$ is said to solve the extended Skorokhod problem for given $g \in D([0,\infty),\mathbb{R})$ if and only if it satisfies the following properties:*

1. *$f(t) = g(t) + h(t) \in [l(t), r(t)]$ for every $t \in [0,\infty)$.*

2. *For every $0 \leq \tilde{t} < t < \infty$,*

$$\begin{aligned} &h(t) - h(\tilde{t}) \geq 0 \text{ if } f(s) < r(s) \text{ for all } s \in (\tilde{t},t], \\ &h(t) - h(\tilde{t}) \leq 0 \text{ if } f(s) > l(s) \text{ for all } s \in (\tilde{t},t]. \end{aligned} \tag{6.2}$$

3. *For every* $0 \leq t < \infty$,

$$h(t) - h(t-) \begin{cases} \geq 0 & \text{for} \quad f(t) = l(t), \\ = 0 & \text{for} \quad l(t) < f(t) < r(t) \\ \leq 0 & \text{for} \quad f(t) = r(t), \end{cases} \tag{6.3}$$

   where we interpret $h(0-)$ *to be 0.*

If (f,h) *is the unique solution to the extended Skorokhod problem on* $[l,r]$ *for* g *then we write* $f = \Gamma^*_{l,r}(g)$, *and refer to* $\Gamma^*_{l,r}$ *as the associated extended Skorokhod map.*

Next, we will see that a solution to the extended Skorokhod problem is a solution to the Skorokhod problem if and only if h is of finite variation:

Proposition 6.3. *Suppose we are given* $l \in D^-([0,\infty),\mathbb{R})$ *and* $r \in D^+([0,\infty),\mathbb{R})$, *such that* $l \leq r$.

1. *Any solution* (f,h) *to the Skorokhod problem on* $[l,r]$ *solves the extended Skorokhod problem.*

2. *A solution* (f,g) *to the extended Skorkhod problem on* $[l,r]$ *such that* h *has finite variation on every bounded interval solves the Skorokhod problem.*

Proof. The proof can be found in [BURDZY et al. 2009, Proposition 2.3]. □

In fact, we can give an easy condition on l and r ensuring the equivalence:

Corollary 6.4. *Suppose we are given* $l \in D^-([0,\infty),\mathbb{R})$ *and* $r \in D^+([0,\infty),\mathbb{R})$, *such that*

$$\inf_{t\geq 0}(r(t) - l(t)) > 0.$$

Any solution (f,g) *to the extended Skorkhod problem on* $[l,r]$ *solves the Skorokhod problem.*

Proof. To apply Proposition 6.3 we have to show that the only difference is that h is of finite variation, to do that we start by defining $\tau_0 := 0$ and for $n \geq 0$

$$\begin{aligned} \tau_{2n+1} &:= \inf\{t \geq \tau_{2n};\, f(t) = r(t)\}, \\ \tau_{2n+2} &:= \inf\{t \geq \tau_{2n+1};\, f(t) = l(t)\}. \end{aligned}$$

This definition ensures that f will touch exactly one of the boundaries l and r on the interval $[\tau_m, \tau_{m+1})$ for each $m \geq 0$. Thus, by (6.2) and (6.3) h will be monotone, and hence of bounded variation on each interval $[\tau_m, \tau_{m+1})$. As $\inf_{t\geq 0}(r(t) - l(t)) > 0$ and $f \in D([0,\infty),\mathbb{R})$ there are finitely many crossovers in each bounded time interval. Consequently, h will have finite variation on each bounded time interval. □

6.1. Existence and uniqueness for the extended Skorokhod problem

Having established the connection between the Skorokhod problem and the extended Skorokhod problem, we turn towards the existence of a solution of the extended Skorokhod problem. As done in Lemma 4.2 we start by proving the existence for step functions. Immediately afterwards we show the uniqueness for general functions implying the uniqueness for step functions.

Proposition 6.5. *Suppose we are given step functions $l \in D^-([0,\infty),\mathbb{R})$ and $r \in D^+([0,\infty),\mathbb{R})$, such that* $\inf_{t\geq 0}(r(t)-l(t)) > 0$. *If g is a step function, then* $\big(g-\Xi_{l,r}(g), -\Xi_{l,r}(g)\big)$*, where*

$$\begin{aligned}\Xi_{l,r}(g)(t) := \max\Bigg(&\big(g(0)-r(0)\big)^+ \wedge \inf_{0\leq s\leq t}\big(g(s)-l(s)\big), \\ &\sup_{0\leq \tilde{t}\leq t}\Big(\big(g(\tilde{t})-r(\tilde{t})\big) \wedge \inf_{\tilde{t}\leq s\leq t}\big(g(s)-l(s)\big)\Big)\Bigg).\end{aligned} \tag{6.4}$$

is a solution to the extended Skorokhod problem on $[l,r]$ for g.

Proof. We denote the set of jumps by $\{\tau_1,...,\tau_n\} \subset \mathbb{N}$ and set $\tau_{n+1} := \infty$. We prove the statement by induction over the number of jumps $n \in \mathbb{N}$.
To start we immediately obtain from the definition of f

$$f(0) = g(0) + \big(g(0)-r(0)\big)^+ \wedge \big(g(0)-l(0)\big) = \begin{cases} l(0) & \text{for } \; g(0) \leq l(0) \\ g(0) & \text{for } \; l(0) \leq g(0) \leq r(0) \\ r(0) & \text{for } \; r(0) \leq g(0). \end{cases}$$

and thus, $f(t) \in [l(t), r(t)]$ for $0 \leq t < \tau_1$, as f is constant on $[0,\tau_1)$. Moreover, h is constant on $[0,\tau_1)$ implying (6.2) and as

$$h(0) = f(0)-g(0) = \begin{cases} l(0)-g(0) \geq 0 & \text{for } \; g(0) \leq l(0) = f(0) \\ 0 & \text{for } \; l(0) \leq g(0) = f(0) \leq r(0) \\ r(0)-g(0) \leq 0 & \text{for } \; f(0) = r(0) \leq g(0), \end{cases}$$

we obtain (6.3).
Next, we assume that (f,h) solves the extended Skorokhod problem on $[l,r]$ for g over the time interval $[0,\tau_m)$ for some $m \in \{1,..,n\}$. A closer examination of $\Xi_{l,r}$ gives us the following alternative representation (for a detailed derivation see [BURDZY et al. 2009, Proposition 2.9])

$$\Xi_{l,r}(g)(t) = \max\big(\Xi_{l,r}(g)(t-), \big(g(t)-r(t)\big)\big) \wedge \big(g(t)-l(t)\big). \tag{6.5}$$

And thus,

$$\begin{aligned} f(t) &= g(t) - \max\big(\Xi_{l,r}(g)(t-), \big(g(t)-r(t)\big)\big) \wedge \big(g(t)-l(t)\big) \\ &= \min\big(g(t)-\Xi_{l,r}(g)(t-), r(t)\big) \vee l(t). \end{aligned}$$

This implies $f(\tau_m) \in [l(\tau_m), r(\tau_m)]$. It is left to show that (6.2) and (6.3) hold for τ_m. As (6.2) and (6.3) holds for $[0,\tau_m)$, (6.3) for τ_m implies (6.2) for $[0,\tau_m]$. We start by considering

$$f(t) = g(t) - \max\Big(\underbrace{\big(g(t-)-f(t-)\big)}_{=\Xi_{l,r}(g)(t-)}, \big(g(t)-r(t)\big)\Big) \wedge \big(g(t)-l(t)\big) \tag{6.6}$$

and assume $f(t) < r(t)$, then we obtain directly

$$\max\big(\big(g(t-)-f(t-)\big), \big(g(t)-r(t)\big)\big) \wedge \big(g(t)-l(t)\big) > g(t)-r(t)$$

We split this implication into

$$g(t-)-f(t-) > g(t)-r(t) \text{ and } g(t)-l(t) > g(t)-r(t)$$

While the second part is trivial, we can use the first one to obtain from (6.6)

$$f(t) = g(t) - \big(g(t-) - f(t-)\big) \wedge \big(g(t) - l(t)\big)$$

and thus, in particular, for $t \geq 0$, such that $f(t) < r(t)$,

$$\begin{aligned} h(t) - h(t-) &= f(t) - g(t) + g(t-) - f(t-) \\ &= g(t-) - f(t-) - \big(g(t-) - f(t-)\big) \wedge \big(g(t) - l(t)\big) \geq 0. \end{aligned} \tag{6.7}$$

We can apply the same argumentation to $t \geq 0$, such that $f(t) > l(t)$, to obtain

$$h(t) - h(t-) \leq 0. \tag{6.8}$$

This gives us (6.3) for τ_m. As this was all that remained to prove, we have shown that (f,h) solves the extended Skorokhod problem on $[l,r]$ for g over the time interval $[0,\tau_m]$.
Lastly, we note that l, r and g are constant on $[\tau_m, \tau_{m+1})$ and therefore as well f and h. This finishes the induction step. □

Proposition 6.6. *Suppose we are given $l \in D^-([0,\infty),\mathbb{R})$ and $r \in D^+([0,\infty),\mathbb{R})$, such that $l \leq r$. For any $g \in D([0,\infty),\mathbb{R})$ there is at most one solution (f,h) to the extended Skorokhod problem on $[l,r]$ for g.*

Proof. Let (f,h) and $(\tilde{f},\tilde{h})$ be two solutions to the extended Skorokhod problem on $[l,r]$ for g. From (6.3) for 0 we obtain directly that $f(0) = \big(g(0) \vee r(0)\big) \wedge l(0)$ and the same for $\tilde{f}$, thus $f(0) = \tilde{f}(0)$.
Now, we assume that $f \neq \tilde{f}$ and without loss of generality $f(T) > \tilde{f}(T)$ for some $T \geq 0$ and define

$$\tau := \sup\{t \in [0,T];\, f(t) \leq \tilde{f}(t)\}. \tag{6.9}$$

As $f(0) = \tilde{f}(0)$, τ is well defined. We distinguish the following two cases:

$\mathbf{f}(\tau) \leq \tilde{\mathbf{f}}(\tau)$: In this case, by the definition of τ we have $l(t) \leq \tilde{f}(t) < f(t) \leq r(t)$, for $t \in (\tau, T]$. This implies that f will not hit l and $\tilde{f}$ will not hit r, by (6.2) we obtain $h(T) - h(\tau) \leq 0$ and $\tilde{h}(T) - \tilde{h}(\tau) \geq 0$. From this we can obtain

$$0 < f(T) - \tilde{f}(T) = h(T) - \tilde{T}(T) \leq h(\tau) - \tilde{h}(\tau) = f(\tau) - \tilde{f}(\tau),$$

contradicting the assumption.

$\mathbf{f}(\tau) > \tilde{\mathbf{f}}(\tau)$: In this case $\tau > 0$ and we obtain from the definition of τ

$$f(\tau-) \leq \tilde{f}(\tau-). \tag{6.10}$$

Moreover, the assumption implies that $f(\tau) > l(\tau)$ and $\tilde{f}(\tau) < r(\tau)$. Thus, (6.3) yields $h(\tau) - h(\tau-) \leq 0$ and $\tilde{h}(\tau) - \tilde{h}(\tau-) \geq 0$ and we can obtain the following contradition to (6.10)

$$0 < f(\tau) - \tilde{f}(\tau) = h(\tau) - \tilde{h}(\tau) \leq h(\tau-) - \tilde{h}(\tau-) = f(\tau-) - \tilde{f}(\tau-).$$

This proves that our initial assumption isn't valid and thus $f = \tilde{f}$ and, therefore, $h = \tilde{h}$. □

As done before, we establish next a closure property for the extended Skorokhod problem. The considered convergence is uniform convergence of $(f_n)_{n\in\mathbb{N}}$ on compacts to f, i.e. for every $T > 0$, $\sup_{0 \leq t \leq T} |f_n(t) - f(t)| \to 0$ as $n \to \infty$.

Proposition 6.7 (Closure property)**.** *Suppose we are given $l_n \in D^-([0,\infty),\mathbb{R})$ as well as $r_n \in D^+([0,\infty),\mathbb{R})$, such that $l_n \leq r_n$, and $g_n \in D([0,\infty),\mathbb{R})$, for each $n \in \mathbb{N}$. Let $l \in D^-([0,\infty),\mathbb{R})$, $r \in D^+([0,\infty),\mathbb{R})$ and $g \in D([0,\infty),\mathbb{R})$ be such that $l_n \to l$, $r_n \to r$ and $g_n \to g$ uniformly on compacts as $n \to \infty$.*
If (f_n,h_n) solves the extended Skorokhod problem on $[l_n,r_n]$ for g_n and $f_n \to f$ uniformly on compacts as $n \to \infty$, then $(f, f-g)$ solves the extended Skorokhod problem on $[l,r]$ for g.

Proof. The proof can be found in [BURDZY et al. 2009, Proposition 2.5]. □

This puts us in the position to state and prove the general existence and uniqueness result for solutions of the extended Skorokhod problem:

Theorem 6.8. *Suppose we are given $l \in D^-([0,\infty),\mathbb{R})$ and $r \in D^+([0,\infty),\mathbb{R})$, such that $l \leq r$. Then for each $g \in D([0,\infty),\mathbb{R})$, there exists a unique pair $(f,h) \in D([0,\infty),\mathbb{R}) \times D([0,\infty),\mathbb{R})$ that solves the extended Skorokhod problem on $[l,r]$ for g admitting the following representation:*

$$\Gamma^*_{l,r}(g) = g - \Xi_{l,r}(g), \tag{6.11}$$

where $\Xi_{l,r}$ is defined as in (6.4)*.*
Furthermore, the map $(l,r,g) \mapsto \Gamma^*_{l,r}$ is a continuous map on $D^-([0,\infty),\mathbb{R}) \times D^+([0,\infty),\mathbb{R}) \times D([0,\infty),\mathbb{R})$ with respect to the topology of uniform convergence on compact sets.

Proof. For $l \in D^-([0,\infty),\mathbb{R})$ and $r \in D^+([0,\infty),\mathbb{R})$ such that $l \leq r$ and $g \in D([0,\infty),\mathbb{R})$ we can find sequences $(l_n)_{n\in\mathbb{N}}$ in $D^-([0,\infty),\mathbb{R})$, $(r_n)_{n\in\mathbb{N}}$ in $D^+([0,\infty),\mathbb{R})$ and $(g_n)_{n\in\mathbb{N}}$ in $D([0,\infty),\mathbb{R})$ such that each element is piecewise constant with a finite number of jumps and $l_n \to l$, $r_n \to r$ and $g_n \to g$ uniformly on compacts as $n \to \infty$.
By Proposition 6.5, $(g_n - \Xi_{l,r}(g_n), -\Xi_{l,r}(g_n))$ solves the extended Skorokhod problem on $[l_n,r_n]$ for g_n, for each $n \in \mathbb{N}$. Since convergence on compacts is preserved under the operations used in the definition of $(l,r,g) \mapsto \Xi_{l,r}(g)$ we obtain the convergence on compacts for $(g_n - \Xi_{l,r}(g_n), -\Xi_{l,r}(g_n))$. This enables us to apply Proposition 6.7 and we obtain a solution $(g - \Xi_{l,r}(g), -\Xi_{l,r}(g))$ to the extended Skorokhod problem on $[l,r]$ of g, this solutions is unique due to Proposition 6.6. In particular, the map $(l,r,g) \mapsto \Gamma^*_{l,r}$ is well-defined and continuous with respect to the topology of uniform convergence. □

6.2. Reflected Brownian motion on time-dependent intervals

This establishes the deterministic case and we can carry this results over to stochastic processes. We start by introducing a complete probability space $(\Omega,\mathscr{A},\mathbb{P})$ and a filtration $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$ fulfilling the usual conditions.

Definition 6.9. *For $l \in D^-([0,\infty),\mathbb{R})$ and $r \in D^+([0,\infty),\mathbb{R})$, such that $l \leq r$, and an $\mathscr{F}$-adapted Brownian motion W, we define the reflected Brownian motion X on $[l,r]$ starting in $x \in \mathbb{R}$ by*

$$X = \Gamma^*_{l,r}(x+W).$$

Moreover, let Λ be the unique process such that, for almost all $\omega \in \Omega$, $(X(\omega),\Lambda(\omega))$ solves the extended Skorokhod problem on $[l,r]$ for $x+W(\omega)$. Λ is called the local time of X on the boundary of $[l,r]$.

As $\Gamma^*_{l,r}(g)(t)$ depends only only on $\{l(s), r(s), g(s); s \in [0,t]\}$, we obtain directly that X is adapted to the filtration $\mathscr{F}$. Remembering the proof of Corollary 6.4, we have that the reflected Brownian motion X is a semimartingale, if $\inf_{t\geq 0}(r(t) - l(t)) > 0$. For a more thorough theory about the variation of Λ we refer to [BURDZY et al. 2009, Section 4].

When we compare the definition of the local time Λ with an adaption of the one given in

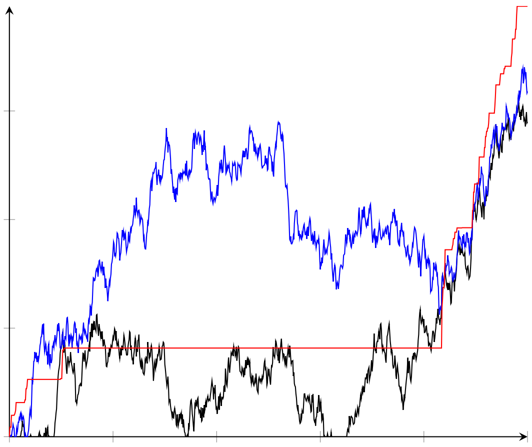

Figure 6.1.: The picture shows a reflected standard Brownian motion —— on $[\max(W,0),\infty)$, for a standard Brownian motion W, and its local time ——.

Chapter 3:

$$\tilde{\Lambda}_t := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \lambda\{s \in [0,t]; X_s < l(s) + \varepsilon \text{ or } X_s > r(s) - \varepsilon\} \tag{6.12}$$

it is clear from the uniqueness that for constant $l = 0$ and $r = \infty$, i.e., the case we considered in Chapter 3 that the two definitions coincide. This does not hold in general, it has been shown that for a Brownian path l independent of W and $r = \infty$ we have $\Lambda = 2\tilde{\Lambda}$ (see [BURDZY and NUALART 2002, Remark 3.1]). This implies that when piecing together a boundary l that looks Brownian on some intervals and is flat otherwise the ratio of Λ to $\tilde{\Lambda}$ varies in time.

7. Conclusions and Perspectives

We want to finish by summing up the results we have obtained. We have proven the existence and, assuming Lipschitz-continuous coefficients, the uniqueness of one-dimensional reflected Brownian motion on the half-line$[0,\infty)$ in Chapter 3 and were able to generalize these results to time-dependent intervals $[a,b]$, with $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$. In the time-independent setting we have seen that the reflection function is just the local time at the boundary. For the time-dependent setting this comes by definition and we pointed out the connection between the definition given in Chapter 3 and Chapter 6.

In Chapter 4 we carried the results over to time-independent multi-dimensional sets, receiving a thorough theory for convex sets and sets with “smooth“ boundaries when the reflection is along a normal vector on the surface. For many applications this limitation to normal vector is to restrictive and we assembled some results for reflection along oblique vectors on the surface in Chapter 5.

As we will see, this puts us in the position to see that there exists a weak solution to the optimal investment and endowment problem we introduced in Chapter 1. We observe that given proportional transaction costs, the direction of reflection on the surface is constant on both sides of the wedge and therefore it is easy to obtain a continuous map $Q : G^{\mathscr{N}} \to \mathbb{R}^d$ satisfying the conditions in Theorem 5.9 by taking the matrix rotating the normal vector onto the direction of reflection. Moreover, the constant directions of reflection on each side of the wedge ensure that the additional conditions in Theorem 5.9 hold and we obtain the existence of a weak solution. For more details concerning this see [DAVIS and NORMAN 1990].

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A. Appendix

A.1. Appendix to Chapter 3

Theorem A.1 (Fatou's Lemma)**.** *Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and $(f_n)_{n\in\mathbb{N}}$ be a sequence of $\mathscr{A}$-measurable functions. If there exists an $h \in L^1(\Omega)$ such that for all $n \in \mathbb{N}$ $f_n \geq h$, then we have*

$$\int_\Omega \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu$$

Definition A.2 (uniformly integrable)**.** *A family C of random variables is called uniformly integrable if*

$$\forall \varepsilon > 0 \exists K \in [0,\infty) \text{ such that } \mathbb{E}(|X| \cdot 1_{|X|>K}) < \varepsilon, \quad \forall X \in C.$$

Theorem A.3 (Doob's Optional-Sampling Theorem for uniform martingales)**.** *If M is a uniformly integrable martingale and τ as well as $\tilde{\tau}$ are stopping times satisfying $\tilde{\tau} \leq \tau$, then*

$$\mathbb{E}(M_\tau | \mathscr{F}_{\tilde{\tau}}) = M_{\tilde{\tau}}, \text{ a.s.}$$

Proof. The proof can be found in [WILLIAMS 2010, A.14.3 Corollary 2]. □

Theorem A.4 (Implicit Function Theorem)**.** *Let $f : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n, (x,u) \mapsto f(x,u)$ continuously differentiable. For any point $(a,b) \in \mathbb{R}^n \times \mathbb{R}^r$ such that $\det \frac{\partial f}{\partial x}(a,b) \neq 0$, we can solve $f(x,u) = 0$ locally around (a,b), i.e., we can find a function g such that locally $f(x,u) = 0$ if and only if $g(u) = x$. Moreover, we have*

$$g'(b) = -\left(\frac{\partial f}{\partial x}(a,b)\right)^{-1} \cdot \frac{\partial f}{\partial u}(a,b).$$

A.2. Appendix to Chapter 4

Theorem A.5 (Hölder's Inequality)**.** *Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$, then, for $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, we have $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \cdot \|g\|_{L^q}$$

Theorem A.6 (Doob's Maximal Inequality)**.** *Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space equipped with a filtration $\mathscr{F}$ and X an $\mathscr{F}$-submartingale with right-continuous path, then we have for any $p > 1$*

$$\mathbb{E}\left(\sup_{s\leq t} W_s\right)^p \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}(W_t^p), \text{ for } t \geq 0.$$

Proof. The proof can be found in [KARATZAS and SHREVE 2000, Theorem 1.3.8]. □

Theorem A.7 (Markov's Inequality)**.** *Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space and X a real-valued random variable such that $\mathbb{E}|X| < \infty$.*
For any constant $\varepsilon > 0$, it is obvious that $\mathbb{E}|X| \geq \varepsilon \cdot \mathbb{P}(|X| > \varepsilon)$, and this implies directly Markov's Inequality

$$\mathbb{P}(|X| > \varepsilon) \leq \frac{\mathbb{E}|X|}{\varepsilon}$$

Theorem A.8 (Borel-Cantelli Lemma)**.** *Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space and $(E_n)_{n\in\mathbb{N}}$ be a sequence of events in $\mathscr{A}$, if $\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty$, then $\mathbb{P}(limsup_{n\to\infty} E_n) = 0$.*

Proof. As $(\sum_{n=1}^{N} \mathbb{P}(E_n))_{N\in\mathbb{N}}$ is bounded and monotone, it converges, implying that the remainder $\sum_{n=N}^{\infty} \mathbb{P}(E_n))_{N\in\mathbb{N}}$ converges to 0. We can use this to calculate

$$\mathbb{P}(\text{limsup}_{n\to\infty} E_n) = \mathbb{P}(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} E_n) \leq \inf_{N\in\mathbb{N}} \mathbb{P}(\bigcup_{n\geq N} E_n) \leq \inf_{N\in\mathbb{N}} \sum_{n\geq N}^{\infty} \mathbb{P}(E_n) = 0.$$

□

Definition A.9 (tightness)**.** *Let $(\Omega, \mathscr{A})$ be a measurable space. A sequence $(\mathbb{P}_n)_{n\in\mathbb{N}}$ of probability measures on $(\Omega, \mathscr{A})$ is called tight if $\forall \varepsilon > 0 \exists K \in \mathscr{A}$ compact such that $\mathbb{P}_n(K) \geq 1 - \varepsilon$ for all $n \in \mathbb{N}$.*

Theorem A.10 (Prohorov's Theorem)**.** *Let $(\Omega, \mathscr{A})$ be a measurable space. If a sequence $(\mathbb{P}_n)_{n\in\mathbb{N}}$ of probability measures is tight, then there is a subsequence $(n_k)_{k\in\mathbb{N}}$ converging weakly to some probability measure on $(\Omega, \mathscr{A})$.*

Theorem A.11 (Skorokhod's Representation Theorem)**.** *Let $(\Omega, \mathscr{A})$ be a measure space. If a sequence $(\mathbb{P}_n)_{n\in\mathbb{N}}$ of probability measures on $(\Omega, \mathscr{A})$ converging weakly to a probability measure $\mathbb{P}$ on $(\Omega, \mathscr{A})$, then there exists a probability space $(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{\mathbb{P}})$ carrying a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables and a random variable X such that*

1. *$(\mathbb{P}_n)$ is the probability measure associated to X_n, for each $n \in \mathbb{N}$, and $\mathbb{P}$ is the probability measure associated to X.*
2. *$X_n \to X$ almost surely as $n \to \infty$.*

Proof. The proof can be found in [WILLIAMS 2010, 17.3 Skorokhod representation]. □

Declaration

I hereby declare that this thesis is entirely the result of my own work except where otherwise indicated. I have only used the resources given in the list of references.

Sören Sanders Kaiserslautern, June 13, 2012