Transcendental Brauer-Manin obstructions on Kummer surfaces

In 1970, Manin observed that the Brauer group $Br(X)$ of a variety X over a number field K can obstruct the Hasse principle on X. In other words, the lack of a K-rational point on X despite the existence of points everywhere locally is sometimes explained by non-trivial elements in $Br(X)$. Since Manin's observation, the Brauer group and the related obstructions have been the subject of a great deal of research.
The 'algebraic' part of the Brauer group is the part which becomes trivial upon base change to an algebraic closure of K. It is generally easier to handle than the remaining 'transcendental' part and a substantial portion of the literature is devoted to its study. The transcendental part of the Brauer group is generally more mysterious, but it is known to have arithmetic importance – it can obstruct the Hasse principle and weak approximation.
I will use class field theory together with results of Skorobogatov and Zarhin to compute the transcendental part of the Brauer group for certain Kummer surfaces related to products of elliptic curves with complex multiplication. I will give examples where there is no Brauer-Manin obstruction coming from the algebraic part of the Brauer group but a transcendental Brauer class causes a failure of weak approximation.