Cheeger's inequality revisited

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In this talk, I presented the ideas and results from the preprint 'The first eigenvalue of the Laplacian, isoperimetric constants, and the Max Flow Min Cut Theorem', arXiv.org:math.DG/0506243.

Cheeger's inequality gives a lower bound for the first eigenvalue of the Dirichlet Laplacian on a compact Riemannian manifold Ω with boundary (assumed Lipschitz),

$$\lambda_\Omega = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}, \tag{1}$$

in terms of 'Cheeger's constant'

$$h_\Omega = \inf_{S \subset \Omega} \frac{|\partial S|}{|S|}. \tag{2}$$

Cheeger proved [**?**]

$$\lambda_\Omega \geq h_\Omega^2/4. \tag{3}$$

We consider the problem of estimating h_Ω from below. First, it is an immediate consequence of Green's formula that, for any number h and vector field V on Ω satisfying the pointwise estimates

$$|V| \leq 1 \tag{4}$$

$$\operatorname{div} V \geq h, \tag{5}$$

one has $h_\Omega \geq h$. This simple fact seems to be little known in the geometric analysis community.

It is a remarkable fact that this estimate is sharp:

Theorem: We have

$$h_\Omega = \sup\{h : \exists V \text{ satisfying } (\mathbf{??}), (\mathbf{??})\},$$

where the supremum is taken over smooth vector fields V on Ω.

A maximizer exists of regularity $V \in L^\infty, \operatorname{div} V \in L^2$.

This theorem may be regarded as a continuous version of the classical Max Flow Min Cut Theorem for networks. In the case of Euclidean domains Ω, it was first proved by Strang [**?**] in two dimensions and by Nozawa [**?**] in general. Their proofs carry over immediately to the case of Riemannian manifolds. Essentially, the supremum in the theorem is regarded as a convex optimization problem. By standard theory, it has a dual problem of the same value, which turns out to be

$$\text{Minimize } \frac{\int_\Omega |\nabla \phi|}{\int_\Omega \phi}, \quad \text{subject to } \phi \geq 0, \phi_{\partial\Omega = 0}. \tag{6}$$

Now the Cavalieri principle and the coarea formula easily imply that this infimum doesn't change if ϕ is restricted to be a characteristic function of a subset $S \subset \Omega$, in which case the numerator has to be interpreted as $|\partial S|$. Therefore, this mimimum is simply h_Ω, and this proves the theorem.

The characterization of Cheeger's constant in terms of vector fields also gives a new approach to the inequality

$$\lambda_\Omega \geq \frac{1}{4\rho_\Omega^2}, \tag{7}$$

(where ρ_Ω is the inradius and Ω is assumed to be a plane simply connected domain). This inequality is usually attributed to Osserman [**?**], but in fact it was first proved by E. Makai [**?**].

Problem: In the example of a square the Cheeger constant and a minimizing subset S can be found explicitly (round off the corners by quarter circles, optimize over their radius). However, there does not seem to be a simple formula for the optimal vector field in the theorem.

References

[1] Cheeger, J., *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in analysis, Princeton Univ. Press, Princeton, New Jersey, 1970, 195- 199.

[2] Makai, E., *A lower estimation of the principal frequencies of simply connected membranes*, Acta Math. Acad. Sci. Hungar., **16** (1965),319-323.

[3] Nozawa, Ryōhei, *Max-flow min-cut theorem in an anisotropic network*, Osaka J. Math., **27**(1990),805-842.

[4] Osserman, R., *A note on Hayman's theorem on the bass note of a drum*, Comment. Math. Helv., **52**(1977),545-555.

[5] Strang, Gilbert, *Maximal flow through a domain*, Math. Programming, **26** (1983),123-143.