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Asymptotics of the first nodal line of a convex domain

Daniel Grieser, David Jerison

Massachusetts Institute of Technology, Department of Mathematics, Cambridge, MA 02139, USA

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1 Introduction

Let Ω be a bounded, convex domain in $\mathbb{R}^2$. Let $0 < \lambda_1 < \lambda_2 \leqq \lambda_3 \leqq \cdots$ be the eigenvalues of the Laplace operator on Ω with Dirichlet boundary conditions. Let u be a second eigenfunction, i.e.,

$$(\Delta + \lambda_2)u = 0 \text{ on } \Omega \quad u = 0 \text{ on } \partial\Omega . \tag{1}$$

The object of this paper is to show that the nodal line,

$$\Lambda = \{x \in \Omega : u(x) = 0\} ,$$

is close to a straight line when the eccentricity of Ω is large.

In order to state a precise estimate, we will normalize Ω to fit inside an $N \times 1$ rectangle whose orientation is chosen carefully. Namely, rotate Ω so that its projection on the y-axis has the shortest possible length, and then dilate so that this projection has length 1. Denote by N the length of the projection of Ω on the x-axis. Then $N \geqq 1$, and N is essentially the diameter of Ω.

Theorem 1 *With the normalization above, there is an absolute constant C_0 such that the width of the nodal line Λ is at most C_0/N. In other words, there exists x_0 such that*

$$(x, y) \in \Lambda \Rightarrow |x - x_0| < C_0/N .$$

We will also prove a pointwise bound on the slope of Λ, at least away from the boundary.

Theorem 2 *Let $\eta = (\eta_1, \eta_2)$ be a unit vector tangent to Λ at the point (x, y). For any $\varepsilon > 0$ there is a number C, only depending on ε, such that if (x, y)*

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has distance at least ε *from the boundary then*

$$|\eta_1| \leqq C/N \, .$$

These bounds are optimal as we will show shortly. Initially we were able to prove weaker bounds on the rate at which the nodal line tends to a straight line, and these were mentioned in [J3]. The best possible rate given here was announced in [GJ]. In that announcement one can also find speculation about the optimal rate in higher dimensions.

We will now show that the bounds in Theorems 1 and 2 are best possible using the example of a circular sector. The method of separation of variables implies that the eigenfunctions of a sector are given by products of trigonometric functions with Bessel functions and that the first nodal line is a circular arc. It is then easy to compute explicitly the extent to which the circular arc deviates from a straight line.

Consider the function given in polar coordinates by

$$u = J_n(r)\sin(n\theta)$$

where J_n is the nth Bessel function. It is well known that the zeros $r_1 < r_2 < \cdots$ of the Bessel function have the asymptotic expansions of the form $r_i = n + c_i n^{1/3} + O(n^{-1/3})$ where $0 < c_1 < c_2 < \cdots$. Define a sector in polar coordinates by

$$\Omega_n = \{(r,\theta) : 0 < r < r_2,\ 0 < \theta < \pi/n\} \, .$$

Then u is the second eigenfunction for Ω_n, and the nodal line is the circular arc where $r = r_1$. This domain is not quite normalized in the same way as in the statement of the theorem. Its projection on the y-axis is smallest, but has length $r_2 \sin(\pi/n)$ which tends to π as $n \to \infty$, rather than being equal to 1. Thus it is contained (modulo lower order terms as $n \to \infty$) in an $n \times \pi$ rectangle. The projection of Λ onto the x-axis has length $r_1(1 - \cos(\pi/n))$ which is asymptotic to $\pi^2/2n$ as $n \to \infty$. Rescaling to the normalizations of the theorem shows that the best constant C_0 in Theorem 1 must be greater than or equal to $1/2$. Numerical evidence from a program that computes eigenfunctions for polygons developed by Toby Driscoll [D] indicates that the constant C_0 of Theorem 1 is less than 1 at least when $N \geqq 3$. We have not attempted to give a reasonable rigorous bound on C_0 because we use bounds of [J] which are already very poor.

In general, the nodal line meets the boundary at a right angle so that its width should be at best comparable to the difference in slopes of the upper and lower boundaries of Ω. For $N \times 1$ convex domains, the worst case is a triangle (or the very similar circular sector) with a difference of slopes of $1/N$. In the case of a rectangle the difference of slopes is zero and the nodal line is exactly straight. The much more detailed theorem, stated in the next section, says, roughly speaking, that the width of the nodal line is bounded by the difference between the slopes, modulo exponentially small errors coming from parts of the domain far away from Λ.

Note further that the choice of normalizing rotation is not unique. In the case of the circular sector the coordinates were chosen so that the projection onto the y-axis had the smallest length. But that is also true of the projection perpendicular to the other straight boundary segment. The error made in projection of a vertical unit line in one coordinate system onto the x-axis in the other coordinate system is comparable to $1/N$. The convex regions for which a more precise statement can be made are ones for which the difference in slopes between the top and bottom boundaries enforces a narrower range of choices for the vertical direction.

Let us now discuss the location of the nodal line, in other words, how one finds the value of the number x_0 in the theorem. This should be compared with the earlier result of [J], in which it is proved that the nodal line has a bounded diameter, independent of N. The present result is much more precise, but it comes at a price. Here we obtain the location of Λ to within $O(1/N)$ using knowledge of eigenvalues of convex two-dimensional domains. In [J], one obtains the bound $O(1)$, but one can compute the approximate location of Λ in terms of eigenvalues of an *ordinary* differential equation.

Here is one way to find x_0. Because the first Dirichlet eigenvalue of a region decreases as the region gets larger, there is a unique vertical segment $\{(x, y) \in \Omega : x = x_1\}$ dividing Ω into two regions with equal first eigenvalue. The min-max principle implies that Λ divides Ω into two regions with equal first eigenvalue and that these are the smallest possible eigenvalues for such a partition. It follows that Λ must intersect the vertical segment. In other words, the number x_0 in the theorem can be taken to be the same as this equipartition value x_1. Thus one can compute the location of the nodal line to within an accuracy of $O(1/N)$ if one can compute the lowest eigenvalue of convex planar regions formed by slicing the original domain by a vertical cut. (The order of accuracy to which the eigenvalues need to be specified will be made precise in terms of a parameter other than the diameter which will be described below.) In [J] the same rotation normalization is used and projection of the nodal line onto the x-axis is located within a bounded distance of a number x_2, which is the zero of the second eigenfunction of an ordinary differential equation. The number x_2 can be determined if one knows how to compute the lowest Dirichlet eigenvalue on an interval for an ordinary differential operator of the form $-(d/dx)^2 + V$ with a potential V that is explicitly determined by the shape of Ω. ($V = \pi^2/h(x)^2$ where h is the width function defined below.)

The proofs given here depend on the results of [J], but the bound is much more precise than in [J]. Moreover, it may be possible, using the new techniques introduced here, to simplify the proof of [J]. A key ingredient is a well-known technique using differential inequalities due to Carleman. Here it is used in a new way to estimate eigenfunctions rather than harmonic functions.

The paper is organized as follows. In Section 2 the main theorem, Theorem 3, is stated, and an outline of the proof is given. This theorem is a sharper version of Theorems 1 and 2 in which bounds are expressed in terms of the slope of the upper and lower boundaries of the domain. The details of the proof of the main theorem can be found in Sects. 3 to 6. Section

2 also presents in Theorem 4 a precise statement of estimates of the second eigenfunction which can be derived from the results of [J]. The proof of this theorem is given in Sects. 7 and 8.

2 Main results and an outline of the proof

Recall that we have normalized the domain Ω by supposing that the projection on the y-axis has the smallest length and that length is 1. Ω is then given by

$$\Omega = \{(x,y) \in \mathbb{R}^2 : f_1(x) < y < f_2(x),\ x \in [a,b]\}$$

with $N = b - a$, and f_1, f_2 are convex and concave functions on $[a,b]$ respectively. Denote the width of Ω at x by $h(x)$,

$$0 \leqq h := f_2 - f_1 \leqq 1 \ . \tag{2}$$

Introduce the slope

$$S(x) = \max\{|f_1'(x)|, |f_2'(x)|\} \qquad S_1(x) = \min\{S(x), 1/N\} \tag{3}$$

(defined almost everywhere). As above, define x_1 as the unique number such that the regions $\{(x,y) \in \Omega : x < x_1\}$ and $\{(x,y) \in \Omega : x > x_1\}$ have the same first Dirichlet eigenvalue.

As explained in [J], the essential length scale governing the shape of the eigenfunction u and the size of the eigenvalues is not N, but a number L, defined as follows. L is the length of the longest interval I such that

$$h(x) \geqq 1 - \frac{1}{L^2} \tag{4}$$

on I. The number L is approximately the length of the rectangle contained in Ω with lowest first eigenvalue, a kind of optimal inscribed rectangle. The eigenfunction u is supported near this rectangle and decays at an exponential rate $e^{-c|x-x_1|/L}$ outside the rectangle. (This will not be proved here, but it can be proved using Carleman-type arguments that are much simpler than the ones given below.) It is easy to show that

$$N^{1/3} \leqq L \leqq N \ ,$$

The main theorem is as follows.

Theorem 3 *The width of the nodal line is at most*

$$W = C_0 \max_{x \in [a,b]} e^{-c|x-x_1|} S_1(x) + e^{-cL} \ .$$

In other words, there is a number x_1 and absolute constants C_0 and $c > 0$ such that

$$(x,y) \in \Lambda \Rightarrow |x - x_1| < W \ .$$

Thus, one can replace $1/N$ *in Theorem* 1 *with the smaller bound* W. (*Indeed*, $W \leqq C/L^3 \leqq C/N$.) *Likewise, one can replace* $1/N$ *in Theorem* 2 *with the smaller bound* W.

This theorem implies Theorems 1 and 2, and it is much sharper in the case in which the upper and lower boundary of Ω are very close to horizontal near the nodal line. In particular, it gives the correct exponentially small bound in the case of a rectangle that is changed at the ends by an arbitrary perturbation of at most unit size. The proof of this is simpler than the general case, and it is carried out in [GJ].

Basic bounds on eigenvalues asymptotically correct to order of magnitude are as follows:

$$\pi^2 \leqq \lambda_1 < \lambda_2 < \pi^2 + 10\pi^2 L^{-2} . \tag{5}$$

This is easy to check. The lower bound holds because $h(x) \leqq 1$, and the upper bound can be proved using test functions in two rectangles of length $L/2$ in the variational characterization of λ_2. (See [J].)

The proof of Theorem 3 begins with the proof that the positive and negative parts $u_\pm$ of u resemble bumps of length L. This is made precise in the following theorem, which we will deduce from the main results of [J] in Sects. 7 and 8.

Theorem 4 *Let u be the second eigenfunction for Ω. Replace u by a suitable constant multiple to obtain the normalization* $\max |u| = 1$. *There are absolute constants $c_1 > 0$, C_1, and L_1 such that if $L \geqq L_1$, then there are numbers A and B with $B-A = L/20$ such that* (4) *holds on* $[A,B]$ *and whenever* (x_0, y_0) *belongs to* Λ, *one has*

$$A + L/50 < x_0 < B - L/50 , \tag{6}$$

$$S(x) \leqq C_1/L^3 \quad \text{for } A \leqq x \leqq B . \tag{7}$$

Moreover, u satisfies (*after multiplication by* ± 1)

$$u(B, 1/2) - u(A, 1/2) \geqq c_1 \tag{8}$$

and

$$|u(x,y)| + |\nabla u(x,y)| < C_1(1 + |x - x_0|)/L \quad \text{for all } (x,y) . \tag{9}$$

From now on, attention will be restricted to the part of Ω between $x = A$ and $x = B$. We proceed to give an outline of the proof of Theorem 3. As in [J], the main principle is to approximate the PDE problem by an ODE problem, for which good estimates are easier to obtain. The idea is that the eigenfunction u is well approximated by some function of the form

$$\tilde{u}(x,y) = e(x,y)\psi(x) \quad \text{with} \tag{10}$$

$$e(x,y) = \sqrt{2/h(x)}\,\sin\alpha(x,y) , \tag{11}$$

$$\alpha(x,y) = \pi\frac{y - f_1(x)}{h(x)} . \tag{12}$$

The reason why this is a reasonable strategy is that $e(x, y)$ is the (normalized) sine function in y representing the lowest frequency for the vertical interval in x. In [J], ψ was defined as an eigenfunction for the ordinary differential operator $(-d/dx)^2 + \pi^2/h(x)^2$, which is the natural differential operator that arises by separation of variables. Here we define $\psi(x)$ in terms of u itself as the lowest frequency component of u in the y-direction, i.e.,

$$\psi(x) = \int_{f_1(x)}^{f_2(x)} e(x, y)u(x, y)\,dy\,. \tag{13}$$

Then ψ satisfies an ODE,

$$\psi'' = -\left(\lambda_2 - \frac{\pi^2}{h^2}\right)\psi + \sigma$$

with an "error term" $\sigma(x)$ coming from the x-dependence of e. In the case of the rectangle there is a complete separation of variables, e is independent of x, and $\sigma = 0$. In Section 3 we will show using $(8, 5, 2)$ that ψ' is positive of order $1/L$ throughout $[A, B]$. It follows that ψ has a unique zero x_0 in $[A, B]$ at a distance atleast cL from A and B and

$$|\psi(x)| \geqq c|x - x_0|/L\,.$$

This lower bound says that $|\tilde{u}(x, y)|$ gets larger the farther x is from x_0. The theorem then follows from an estimate on the error

$$v = u - \tilde{u} \tag{14}$$

of the form

$$|v(x, y)| \leqq e(x, y)W/L\,. \tag{15}$$

Namely, if $(x, y) \in \Lambda$ then $v(x, y) = -\tilde{u}(x, y)$ and so

$$e(x, y)c|x - x_0|/L < e(x, y)|\psi(x)| \leqq e(x, y)W/L \tag{16}$$

which implies the first part of Theorem 3, after adjustment of the constants. Actually, because we only have a bound on the integral of the curvature, h'', rather than a uniform pointwise bound, we only obtain a slightly weaker estimate on v at a distance to the boundary that is exponentially small in W; a separate argument is then needed within that distance.

To prove the error estimate (15), we use the Carleman method. Introduce the quantity

$$\phi(x) = \int_{f_1(x)}^{f_2(x)} v^2\,dy\,. \tag{17}$$

Then the differential equations for ψ and u imply that ϕ satisfies a differential inequality of the form

$$\phi'' \geqq \phi - \beta\sqrt{\phi}\,.$$

This is a convexity condition on ϕ that forces ϕ to be small near x_0. If β were zero, then the solution would be bounded above by a hyperbolic cosine. Since ϕ is bounded at the endpoints A and B, it must be exponentially small near x_0. In generally β is not zero, but it is bounded in terms of $S_1(x)$ and L and this gives the correct bound on ϕ. The novelty here is in the possibility of obtaining a coefficient as large as 1 on ϕ on the right-hand side. The reason it can be obtained is that we have subtracted the lowest "vertical frequency" of v, so that its next frequency is $4\pi^2/h^2$, which exceeds λ_2 by almost $3\pi^2$. (See Section 4 for details.) The Carleman technique is usually used directly on the square of the solution to a differential equation instead of one from which we have subtracted the first Fourier component.

Finally, the bounds on ϕ and estimates of Green's function are used to prove the pointwise bounds on v of (15), in Section 5. As we mentioned earlier, this pointwise estimate fails for points extremely close to the boundary, and a separate argument is needed there. That argument is given in Section 6. The slope estimate, which forms the second part of Theorem 3, is proved at the end of Section 6.

3 The projection onto the first vertical frequency

We proceed to analyze $\psi(x)$, defined by (13). From now on we write $\lambda = \lambda_2$. In the following, first derivatives for f_1 and f_2 exist almost everywhere, and second derivatives are understood in the sense of distributions. We denote $e_x = (\partial/\partial x)e$ etc.

Lemma 1 *If $A \leqq x \leqq B$ then*

$$|e_x(x,y)| \leqq CS(x) \quad \textit{and} \tag{18}$$

$$|e_{xx}(x,y)| \leqq C(S(x) + |h''(x)|)\,. \tag{19}$$

Proof. One calculates first that these bounds are satisfied with e replaced by α. In fact, (4) implies, in particular, that h is bounded below by $1/2$ on $[A,B]$. Therefore,

$$|\alpha_x| = \pi|(y - f_1)h'/h^2 + f_1'/h| \leqq CS(x)$$

and hence

$$|e_x| = |(-\sqrt{2}/2)h^{-3/2}h' \sin\alpha + \sqrt{2}h^{-1/2}\alpha_x \cos\alpha| \leqq CS(x)\,.$$

Convexity implies $|h''| = |f_1''|+|f_2''|$. The second order derivative e_{xx} is actually bounded by a multiple of $|h''| + S(x)^2$. But (7) implies $S(x)^2 \leqq CS(x)$. □

In the following calculations, all integrals are from $f_1(x)$ to $f_2(x)$. Let

$$\sigma(x) = \int (2e_x u_x + e_{xx}u)\,dy\,.$$

Then

$$\begin{aligned}\psi''(x) &= \int eu_{xx}\,dy + \sigma(x) = \int e(-\lambda u - u_{yy})\,dy + \sigma(x)\\ &= -\left(\lambda - \frac{\pi^2}{h(x)^2}\right)\psi(x) + \sigma(x)\end{aligned} \tag{20}$$

using integration by parts, $e_{yy} = -\frac{\pi^2}{h^2} e$ and $u = e = 0$ on $\partial\Omega$. Fix a number x_0 so that (x_0, y_0) belongs to Λ. Denote

$$\delta(x) = 1 + |x - x_0| \tag{21}$$

The error σ is estimated using (9, 18, 19) by

$$|\sigma(x)| \leqq C(S(x) + |h''(x)|)\delta(x)/L\ . \tag{22}$$

Lemma 2 *Assume that the function* $\psi(x)$ *satisfies the equation*

$$\psi'' + \rho\psi = \sigma$$

on $[A, B]$, $B - A = L/20$,

$$\psi(A) < 0 < \psi(B) \tag{23}$$

and that the functions $\rho(x), \sigma(x)$ *satisfy*

$$|\rho| \leqq 100/L^2 \tag{24}$$

$$\int_A^B |\sigma| \leqq Q/8 \tag{25}$$

where $Q = (\psi(B) - \psi(A))/(B - A)$. *Then*

$$Q/8 \leqq \psi' \leqq 2Q$$

throughout $[A, B]$.

Proof. Change variables to unit scale by

$$w(t) = \psi(A + (B - A)t)\ .$$

Then w satisfies

$$w'' + Vw = \eta$$

with

$$V(t) = (B - A)^2 \rho(A + (B - A)t) \quad \text{and} \quad \eta(t) = (B - A)^2 \sigma(A + (B - A)t)\ .$$

Let $Q_1 = w(1) - w(0) = (B - A)Q$. Then V and η satisfy

$$|V| \leqq 1/4$$

and

$$\int_0^1 |\eta(t)|\, dt \leqq Q_1/8\ .$$

The conclusion of the lemma can be written

$$Q_1/8 \leqq w'(t) \leqq 2Q_1 \quad \text{for } 0 \leqq t \leqq 1\ .$$

Let w_0, w_1 be solutions to the homogeneous equation (with η replaced by 0) and boundary conditions

$$w_0(0) = -1, \quad w_0(1) = 0$$
$$w_1(0) = 0, \quad w_1(1) = 1\ .$$

We will first prove a corresponding estimate for $w = w_0, w_1$. The upper bound on V and Sturm comparison imply that the distance between any two zeroes of w_0 or w_1 is at least 2π. It follows that neither changes sign in $[0,1]$. Now let $\overline{w} = \sqrt{2}\sin(\pi x/4)$. Then

$$\overline{w}(0) = 0, \quad \overline{w}(1) = 1 \quad \text{and} \quad \overline{w}'' + \left(\frac{\pi}{4}\right)^2 \overline{w} = 0 \, .$$

From $\pi^2/16 > 1/4 \geqq V$ and the generalized maximum principle (see [J], [PW]) it now follows that

$$0 \leqq w_1 \leqq \overline{w} \leqq 1, \qquad w_1'(0) \leqq \overline{w}'(0) = \pi/2\sqrt{2}, \qquad w_1'(1) \geqq \overline{w}'(1) = \pi/4 \, .$$

For $0 \leqq t \leqq 1$ we then get

$$w_1'(t) \geqq w_1'(1) - \left|\int_t^1 V w_1\right| \geqq \pi/4 - 1/4$$

and

$$w_1'(t) \leqq w_1'(0) + \left|\int_0^t V w_1\right| \leqq \pi/2\sqrt{2} + 1/4 \, ,$$

so

$$1/2 \leqq w_1' \leqq 3/2$$

throughout $[0,1]$. The same inequalities hold for w_0' using a similar comparison function.

Let the Wronskian of w_0 and w_1 be $a = w_0 w_1' - w_1 w_0'$. Note that a is constant since the equation has no first derivative term. Now write $w(t) = h(t) + w_2(t)$ with $h(t) = -w(0)w_0(t) + w(1)w_1(t)$. From (23) it now follows that

$$Q_1/2 \leqq h'(t) \leqq 3Q_1/2$$

Because $h(1) = w(1), h(0) = w(0)$, and $h'' + Vh = 0$, we have $w_2(0) = w_2(1) = 0$ and $w_2'' + Vw_2 = \eta$. Therefore

$$w_2(t) = -w_0(t)\int_0^t \frac{\eta(\tau)w_1(\tau)}{a}\,d\tau - w_1(t)\int_t^1 \frac{\eta(\tau)w_0(\tau)}{a}\,d\tau$$

Furthermore, we have $a = -w_1'(0) \leqq -1/2$ and

$$w_2'(t) = -w_0'(t)\int_0^t \frac{\eta(\tau)w_1(\tau)}{a}\,d\tau - w_1'(t)\int_t^1 \frac{\eta(\tau)w_0(\tau)}{a}\,d\tau \, ,$$

so we get

$$|w_2'| \leqq 3\int_0^1 |\eta| \, .$$

This implies the lemma.

Let us remark that σ need not be an L^1 function. It suffices if σ is a measure and $\int |\sigma|$ represents the total variation. □

We wish to apply the lemma to ψ in (13), and so we choose

$$\rho = \lambda - \frac{\pi^2}{h^2} \, ,$$

From (2, 4, 5) we get $|\rho| \leqq 100/L^2$. Also, (8) and (9) imply $Q > cL^{-1}$, and (22, 7) give $\int_A^B |\sigma| < CL^{-2}$. (For the latter bound, $|h''|$ may be a measure rather than an L^1 function, but the total variation is bounded by C/L^3, the maximum of S in $[A, B]$.) Therefore, the lemma applies for sufficiently large L. Since clearly $Q \leqq CL^{-1}$, the conclusion of the lemma then implies that ψ has a unique zero x_0 in $[A, B]$ and that

$$c|x - x_0|/L \leqq |\psi(x)| \leqq C|x - x_0|/L \tag{26}$$

for positive constants c, C. Moreover, the definition of ψ shows that u must change sign on the vertical line $x = x_0$, in other words, (x_0, y_0) belongs to Λ for some y_0.

4 L^2 estimates for the error term

We now turn to the estimation of the error term $v(x, y)$. First, we derive an approximate PDE for $v = u - \tilde{u}$, $\tilde{u}(x, y) = e(x, y)\psi(x)$:

$$\begin{aligned} -(\Delta + \lambda)v = (\Delta + \lambda)\tilde{u} &= (e_{yy} + \lambda e)\psi + e\psi'' + 2e_x\psi' + e_{xx}\psi \\ &= e\sigma + 2e_x\psi' + e_{xx}\psi \ . \end{aligned} \tag{27}$$

Using (20, 22, 18, 19, 26), we find that

$$E = e\sigma + 2e_x\psi' + e_{xx}\psi = (S\delta + |h''|(e\delta + |x - x_0|))O(1/L) \tag{28}$$

with δ defined in (21). In this section, we only use that the right hand side is bounded by $S\delta/L$, on average over unit x-intervals, but the full precision is needed for the pointwise bounds on v in Section 5.

For fixed x, the function $v(x, \cdot)$, defined on the interval $[f_1(x), f_2(x)]$ and vanishing at its endpoints, equals $u(x, \cdot)$ minus its lowest frequency, i.e. its Fourier series is of the form

$$v(x, y) = \sqrt{2/h(x)} \sum_{k=2}^{\infty} v_k(x) \sin k\alpha(x, y) \ .$$

This implies the fundamental inequality (again all integrals from $f_1(x)$ to $f_2(x)$)

$$-\int vv_{yy}\, dy \geqq \frac{4\pi^2}{h^2} \int v^2\, dy \ . \tag{29}$$

For $\phi(x) = \int v^2(x, y)\, dy$ we then get

$$\begin{aligned} \phi'' &= 2\int (v_x^2 + vv_{xx})\, dy \geqq 2\int vv_{xx}\, dy \\ &= 2\int v(-v_{yy} - \lambda v - E)\, dy \\ &\geqq 2\left(4\frac{\pi^2}{h^2} - \lambda\right) \int v^2 dy - 2\int vE\, dy \end{aligned}$$

The main point is that because of the factor 4, (4) and (5) imply $2(4\pi^2/h^2 - \lambda) \geqq 1$. Denote

$$\beta(x) = 2\left(\int_{f_1(x)}^{f_2(x)} E(x,y)^2 dy\right)^{1/2} .$$

Then β is a measure and the Cauchy–Schwarz inequality gives

$$\phi''(x) \geqq \phi(x) - \beta(x)\sqrt{\phi(x)} . \tag{30}$$

Finally, (28) implies that β satisfies

$$\int_x^{x+1} \beta \leqq CL^{-1} \max_{t\in[x,x+1]} S(t)\delta(t) . \tag{31}$$

Our goal is to deduce from this inequality that ϕ must be fairly small near the nodal line, using that it is bounded by a constant within distance cL from it.

If β were a (positive) constant, one solution of the equation

$$\phi'' = \phi - \beta\sqrt{\phi}$$

would be

$$\phi \equiv \beta^2 ,$$

and it is easy to see that if $\phi(0) > 2\beta^2, \phi'(0) \geqq 0$, say, then ϕ increases exponentially for $x > 1$, so that every solution bounded by one in a long interval of length l must be less than

$$C(e^{-cl} + \beta^2)$$

near the center of the interval.

In our case, the proof of a similar bound is a little harder, since $\beta(x)$ can have delta function singularities, and we have only bounds on averages of β.

Lemma 3 *Suppose the absolutely continuous function* ϕ *satisfies* (30) *on* $[0,1]$, *with*

$$\phi(0) \geqq \left(10\int_0^1 \beta\right)^2 , \qquad \phi'(0) \geqq 0 .$$

Then there exists a point $x \in [0,1]$ *satisfying*

$$\phi(x) > \frac{6}{5}\phi(0), \qquad \phi'(x) \geqq 0 .$$

Proof. Let $m(x) = \min\{\phi(t) : 0 \leqq t \leqq x\}$ and $M(x) = \max\{\phi(t) : 0 \leqq t \leqq x\}$, and write $b = \int_0^1 \beta$. Integrating (30), we get

$$\begin{aligned}\phi'(x) &\geqq \int_0^x \phi - \beta\sqrt{\phi} \geqq xm(x) - b\sqrt{M(x)} \\ &\geqq xm(x) - M(x)/10 \end{aligned} \tag{32}$$

since $b \leqq \sqrt{\phi(0)}/10 \leqq \sqrt{M(x)}/10$.

From this we first deduce that

$$m(x) \geqq \frac{9}{10}\phi(0)$$

for all $x \in [0,1]$. To prove this, we assume without loss of generality that x is a "first minimum", i.e. that $m(x) = \phi(x)$. Since ϕ assumes the value $M(x)$ somewhere on $[0,x]$, say at $x_1 < x$, the mean value inequality shows

$$\phi'(x_2) \leqq \frac{\phi(x) - \phi(x_1)}{x - x_1} = \frac{m(x) - M(x)}{x - x_1} \leqq m(x) - M(x)$$

for some $x_2 < x$. From (32) we now get

$$m(x) - M(x) \geqq \phi'(x_2) \geqq -M(x_2)/10 \geqq -M(x)/10 ,$$

and this gives the lower bound for $m(x)$.

To conclude the proof of the lemma, suppose $M(x) \leqq \frac{6}{5}\phi(0)$ for all x. We will derive a contradiction. Then (32) gives

$$\phi'(x) \geqq \left(\frac{9}{10}x - \frac{1}{10}\frac{6}{5}\right)\phi(0) ,$$

and integrating this from 0 to 1 yields

$$M(1) \geqq \phi(1) > 1.3\phi(0) ,$$

a contradiction. Now choose for x any value in $[0,1]$ where ϕ is at its maximum. Then $\phi'(x) \geqq 0$ and this concludes the lemma. □

Lemma 4 *The function* $\phi(x) = \int_{f_1(x)}^{f_2(x)} v^2 dy$ *satisfies the bound*

$$\phi(x) \leqq (W/L)^2$$

for $|x - x_0| < 3$, *where* x_0 *is the zero of* $\psi(x)$ *and* W *is the* '*width*' *defined in Theorem* 3.

Proof. Set $b(x) = \int_x^{x+1} \beta$. Fix a point $\bar{x}$ and an integer M. Assume first that

$$\phi'(\bar{x}) \geqq 0, \quad \phi(\bar{x}) \geqq \max_{t\in[0,M]} \left(\frac{5}{6}\right)^t (10b(\bar{x}+t))^2 . \tag{33}$$

Applying Lemma 3 inductively one can find numbers $\bar{x} = x_1, x_2, \ldots, x_M$ with

$$x_n < x_{n+1} \leqq x_n + 1$$

$$\phi(x_{n+1}) \geqq \frac{6}{5}\phi(x_n)$$

$$\phi'(x_n) \geqq 0$$

for $n = 1, 2, \ldots, M-1$. On the other hand, $|u|$ is bounded by 1, so that $|\phi(x)| \leqq 3$ for all x in $[A,B]$. Therefore, we see that if $[\bar{x}, \bar{x}+M] \subset [A,B]$ then (33) implies

$$\phi(\bar{x}) < 3\left(\frac{5}{6}\right)^{M-1} .$$

In other words, we have shown that if ϕ satisfies (30) on $[\bar{x}, \bar{x}+M] \subset [A,B]$ and $\phi'(\bar{x}) \geqq 0$ then

$$\phi(\bar{x}) < \max_{t\in[0,M]} \left(\frac{5}{6}\right)^{t} (10b(\bar{x}+t))^2 + 3\left(\frac{5}{6}\right)^{M-1} .$$

Recall that (6) says $A + L/50 < x_0 < B - L/50$, so one can apply this estimate with M equal to the integral part of $L/50$ and any $\bar{x}$ near x_0, after a reflection $x \to -x$ if $\phi'(\bar{x}) < 0$. The lemma now follows immediately from (31). Note that the factor $\delta(t)$ in (31) can be absorbed into the exponential term. □

5 From L^2 bounds to pointwise bounds

Let $z_1 = \inf\{x : (x, y) \in \Lambda\} - 2$ and $z_2 = \sup\{x : (x, y) \in \Lambda\} + 2$. Recall that the main theorem of [J] says that $|z_2 - z_1|$ is bounded above by an absolute constant.

Lemma 5 *The error function $v(x, y)$ satisfies*

$$|v(x, y)| \leqq e(x, y)(1 + |x - x_0|\,|\log e(x, y)|)\frac{W}{L}$$

for $z_1 \leqq x \leqq z_2$.

Note that $e(x, y)$ is comparable to the distance of (x, y) to $\partial\Omega$.

The starting point for the proof is to rewrite the equation (27) as Poisson equation and split the inhomogeneous term into various parts. Also, we introduce a cutoff function $\rho(x)$,

$$\rho \in C_0^\infty(z_1 - 1, z_2 + 1), \qquad \rho(x) = 1 \text{ for } x \in \left[z_1 - \frac{1}{2}, z_2 + \frac{1}{2}\right] .$$

Clearly, ρ can be chosen with uniform (independent of N) bounds on all derivatives. In the range $z_1 - 1 \leqq x \leqq z_2 + 1$, we then have, using (28),

$$\begin{aligned} \Delta(\rho v) &= F_1 + F_2 + F_3 + F_4 \\ F_1(x, y) &= (-\lambda + \rho'')v(x, y) + O(S(x)/L) \\ F_2(x, y) &= h''(x)e(x, y)O(1/L) \\ F_3(x, y) &= h''(x)|x - x_0|O(1/L) \\ F_4(x, y) &= 2\rho'(x)v_x(x, y) . \end{aligned} \tag{34}$$

The logarithmic term in Lemma 5 comes only from F_3.

Let

$$\Omega_0 = \{(x, y) \in \Omega : z_1 - 1 < x < z_2 + 1\} .$$

Since ρv vanishes on $\partial\Omega_0$, we can write

$$\rho v = GF_1 + GF_2 + GF_3 + GF_4 \tag{35}$$

where the integral kernel of the operator G is the Green's function for Ω_0. We will prove the bounds in the Lemma for each of the four summands in (35). The point is that the F_i satisfy certain integral estimates with the correct bounds, and that G satisfies corresponding dual estimates with uniform bounds, vanishing at the boundary like e (resp. $e \log e$ for F_3). For F_4 estimates on a derivative of the Green's function are needed, but these are simple since the support of F_4 has positive distance from $[z_1, z_2]$.

Proof. (of Lemma 5)

For a function $F(\xi,\eta)$ on Ω_0 and $1 \leqq p,\ q \leqq \infty$ we will denote

$$\|F\|_{L^p_\xi(L^q_\eta)} = \| \|F\|_{L^q_\eta} \|_{L^p_\xi} = (\int (\int |F(\xi,\eta)|^q d\eta)^{p/q} d\xi)^{1/p}$$

(for $p, q < \infty$, with analogous definitions if p or q is ∞).

Then, Lemma 4 gives

$$\|F_1\|_{L^\infty_\xi(L^2_\eta)} \leqq W/L\ .$$

Regarding F_2 and F_3, we have

$$\int_{z_1-1}^{z_2+1} |h''(\xi)|\, d\xi \leqq W\ .$$

In the term GF_4 we can integrate by parts and get

$$GF_4(x,y) = -2 \int\int_{\Omega_0} \frac{\partial}{\partial \xi}(G(x,y;\xi,\eta)\rho'(\xi))v(\xi,\eta)\, d\eta d\xi\ .$$

Thus, Lemma 5 follows from:

Lemma 6 *The Green's function satisfies the following bounds*:

$$\|G(x,y;\xi,\eta)\|_{L^1_\xi(L^2_\eta)} \leqq Ce(x,y) \tag{36}$$

$$\|G(x,y;\xi,\eta)e(\xi,\eta)\|_{L^\infty_\xi(L^1_\eta)} \leqq Ce(x,y) \tag{37}$$

$$\|G(x,y;\xi,\eta)(\xi - x_0)\|_{L^\infty_\xi(L^1_\eta)} \leqq Ce(x,y)(1 + |x - x_0|\,|\log e(x,y)|) \tag{38}$$

$$|G(x,y;\xi,\eta)| + |G_\xi(x,y;\xi,\eta)| \leqq Ce(x,y) \quad \textit{for } |x-\xi| \geqq 1/2\ . \tag{39}$$

In the proof of this lemma, and also in Sects. 6 and 7, we will make use of the following bounds on the Green's function of a convex domain. Analogous bounds in dimensions 3 and higher were proved in [GW], and in [F] it is remarked that only minor modifications are needed in two dimensions.

Lemma 7 ([F, GW]) *The Green's function $G = G(x,y;\xi,\eta)$ of Ω_0 satisfies the following bounds. Here $D = |x-\xi| + |y-\eta|$.*

$$|G| \leqq Ce(x,y)D^{-1} \tag{40}$$

$$|\nabla_{\xi,\eta} G| \leqq C \min\{e(x,y)D^{-2}, D^{-1}\} \tag{41}$$

$$|\nabla_{x,y}\nabla_{\xi,\eta} G| \leqq CD^{-2}\ . \tag{42}$$

Proof. (of Lemma 6)

Clearly, (36) and (39) follow directly from (40) and (41). However, for the estimates (37) and (38) the bounds in Lemma 7 are too weak. We will prove these estimates for the Green's function

$$G_0(x, y; \xi, \eta) = \frac{1}{4\pi}(\log[(x-\xi)^2 + (y-\eta)^2] - \log[(x-\xi)^2 + (y+\eta)^2])$$

of the half plane $y > 0$, with $x = 0$ and $e(x, y)$ replaced by y, and the integrals extended over $|\xi| < K := 2(z_2 - z_1) + 5$ and $0 \leqq \eta < 2$.

We first show that the estimates for G_0 imply the estimates for G. Fix a point $P = (\bar{x}, \bar{y})$ and a point P' on $\partial\Omega$ with minimal distance to P. We change coordinates so that P' is the origin, the x-axis does not intersect Ω (here the convexity of Ω is essential) and P lies on the positive y-axis. Then clearly $\bar{y}/2 \leqq e(\bar{x}, \bar{y}) \leqq 2\bar{y}$. For fixed (ξ, η), the function $G - G_0$ of (x, y) is then harmonic in Ω_0 and nonnegative on its boundary, since $G_0 \leqq 0$. By the maximum principle it is therefore nonnegative on Ω_0. Also, $G \leqq 0$ by the maximum principle, and therefore $|G| \leqq |G_0|$. In particular,

$$|G(\bar{x}, \bar{y}; \xi, \eta)| \leqq |G_0(\bar{x}, \bar{y}; \xi, \eta)|$$

for all (ξ, η). This proves the claim.

Now we estimate $|G_0| = \frac{1}{4\pi}\log\frac{\xi^2+(\eta+y)^2}{\xi^2+(\eta-y)^2}$ in two ways:

(a) $$|G_0| \leqq \log\frac{\eta + y}{|\eta - y|} \quad \text{and}$$

(b) $$|G_0| = \frac{1}{4\pi}\log\left(1 + \frac{4y\eta}{\xi^2 + (\eta-y)^2}\right) \leqq \frac{y\eta}{\xi^2 + (\eta-y)^2}\,.$$

In order to prove estimate (37), i.e.

$$\int_0^2 |G_0|\eta\, d\eta \leqq Cy \tag{43}$$

for all ξ, we use estimate (a). We are then left with showing

$$\int_0^2 \eta \log\frac{\eta + y}{|\eta - y|}\, d\eta \leqq Cy\,,$$

a simple exercise.

For estimate (38), i.e.

$$|\xi - x_0|\int_0^2 |G_0|\, d\eta \leqq Cy(1 + |x_0|\,|\log y|)$$

for all ξ and x_0, we prove the two estimates

$$|\xi|\int_0^2 |G_0|\, d\eta \leqq Cy \tag{38'}$$

and

$$\int_0^2 |G_0| d\eta \leqq Cy(1 + |\log y|) . \qquad (38'')$$

The former follows directly from estimate (b) and the fact that the integral $\int_{\mathbb{R}} \xi(\xi^2 + \alpha^2)^{-1} d\alpha$ is finite and independent of ξ. Estimate (38″) follows from (a) and the fact

$$\int_0^2 \log \frac{\eta + y}{|\eta - y|} d\eta \leqq Cy(1 + |\log y|) ,$$

another exercise. □

6 Completion of the proof of Theorem 3

Suppose $(x, y) \in \Lambda$. Then $v(x, y) = -\tilde{u}(x, y)$, so (26) and Lemma 5 imply that

$$e(x, y) \frac{|x - x_0|}{L} \leqq e(x, y) \frac{W}{L} + e(x, y)|x - x_0| \frac{W|\log e(x, y)|}{L} ,$$

and this proves the Theorem for $|z_2 - z_1| W| \log e| < 1/2$. The remaining region is the set of points of Ω_0 where the distance from (x, y) to the boundary of Ω is less than $e^{-c/W}$.

Denote $u_+(x, y) = \max\{u(x, y), 0\}, u_-(x, y) = \max\{-u(x, y), 0\}, \Omega_+ = \{(x, y) \in \Omega : u(x, y) > 0\}$ and $\Omega_- = \{(x, y) \in \Omega : u(x, y) < 0\}$. The estimate just proved shows in particular that the rectangle $R_+ = \{(x, y) : x_0 + 1 < x < z_2, 1/4 \leqq y \leqq 3/4\} \subset \Omega_+$ and $R_- = \{(x, y) : z_1 < x < x_0 - 1, 1/4 \leqq y \leqq 3/4\} \subset \Omega_-$. It follows from Harnack's inequality that all values of $u_-(x, 1/2)$ for $z_1 + 1 \leqq x \leqq x_0 - 2$ are comparable and all values of $u_+(x, 1/2)$ for $x_0 + 2 \leqq x \leqq z_2 - 1$ are comparable. Furthermore, the generalized maximum principle and easy barriers of unit width and length $z_2 - z_1$ show that $\max_{\Omega_0} u_+ \approx u_+(x, 1/2)$ for all the values $x_0 + 2 \leqq x \leqq z_2 - 1$ and similarly for u_-.

The following is the "crossover" lemma from [J], [J2]:

Lemma 8 *Suppose that B_1 is a ball of radius $r < 1/2$ centered at (x, y) and B_2 is the concentric ball of radius $2r$. Suppose further that $B \subset \Omega_-$ and $\partial B \cap \Lambda \neq \emptyset$. Then*

$$|u_-(x, y)| \leqq C \max_{B_2} u_+$$

This lemma shows that in a certain sense the control of the maximum crosses the nodal line so that $\max u_\pm$ must be comparable. In order for the control to pass to the boundary one must also use the Carleson lemma. One can show as in [J] and [J2] that as a corollary of Lemma 8, the Harnack inequality and Carleson lemma

$$\max_{\Omega_0} u_- \approx \max_{\Omega_0} u_+ \approx u_-(x_0 - 2, 1/2) \approx u_+(x_0 + 2, 1/2).$$

We already know that if $x \geqq x_0 + W$ and $(x, y) \in \Lambda$, then $y - f_1(x) < t$ or $f_2(x) - y < t$, where $t = e^{-c/W}$. Suppose that there is a point of $(x, y) \in \Lambda$ with $x = x_0 + 5W$ and $y - f_1(x) < t$. We will show that this leads to a contradiction.

Consider the region

$$S = \{(x, y) \in \Omega_- : x_0 + W \leqq x \leqq x_0 + 5W, y < 1/2\}$$

Since this region has a width at most t and a length $4W$, we will be able to prove that u is exponentially small in the middle $x = x_0 + 3W$. One uses the mean value property for eigenfunctions: If v is an eigenfunction on a disk D of radius $r < 1/2\sqrt{\lambda}$ centered at P, then

$$v(P) = \frac{c(r\sqrt{\lambda})}{s(\partial D)} \int_{\partial D} v\, ds$$

where ds represents arclength and $c(r\sqrt{\lambda}) \leqq 2$. (In the case $\lambda = 0$ of a harmonic function the constant c is 1.) If D is a disc of radius $5t$ centered at a point $(x, y) \in S$, and $x_0 + W + 5t < x < x_0 + 5W - 5t$, then u vanishes on the portions of $\partial(D \cap S)$ that are interior to D. One can compare u by the generalized maximum principle to an eigenfunction on D to obtain

$$-u(x, y) \leqq \frac{2}{s(\partial D)} \int_{\partial(D\cap S)} u \leqq \frac{2s((\partial D) \cap S)}{s(\partial D)} \max_{(\partial D)\cap S} u_- \leqq \frac{1}{2} \max_{(\partial D)\cap S} u_-$$

It follows by induction that

$$\max_{S_k} u_- \leqq 2^{-k} \max_{S} u_-$$

where $S_k = S \cap \{(x, y) : x_0 + W + 5kt \leqq x \leqq x_0 + 5W - 5kt\}$. Finally, we choose K largest so that $5Kt \leqq W$. Then $K \approx We^{c/W}$ and

$$\max_{S_K} u_- \leqq 2^{-K} \max_{\Omega_0} u_-$$

In particular, S_K contains all $(x, y) \in S$ such that $x_0 + 2W \leqq x \leqq x_0 + 4W$. Let Z be the point $(x_0 + 3W, f_1(x_0 + 3W) + W/2)$. Then $Z \in \Omega_+$. Let B_1 be the largest disk centered at Z contained in $\bar{\Omega}_+$. The radius of B_1 is less than, but comparable to $W/2$. This disk is tangent to Λ at at least one point and B_2, the concentric double of B_1, satisfies $B_2 \cap \Omega_- \subset S_K$. It follows from Lemma 8 that

$$u(Z) \leqq C2^{-K} \max_{\Omega_0} u_-$$

Next, the Harnack inequality, applied to a chain of balls of radius $2^j W$, $j = 1, \ldots, J$ with $J = C \log(1/W)$, implies that

$$u(x_0 + 2, 1/2) \leqq C^J u(Z) \approx W^{-C} u(Z)$$

Putting all these inequalities together we have

$$\max_{\Omega_0} u_- \leqq C2^{-K} W^{-C} \max_{\Omega_0} u_- ,$$

where C is an absolute constant. Finally, for W sufficiently small, the factor on the right-hand side is less than 1/2 and this contradicts the fact that u_- is nonzero is Ω_0. All other cases lead to a similar contradiction so we have proved that the nodal line has a width bounded by $5W$, and, renaming W, this concludes the proof of the first part of Theorem 3.

We now prove the second part of Theorem 3, i.e. the slope estimate. Let $(\bar{x}, \bar{y}) \in \Lambda$ and assume $e(\bar{x}, \bar{y}) > \varepsilon$. Since the vector (u_x, u_y) is normal to Λ, it is clearly enough to prove that, at $(\bar{x}, \bar{y})$, we have

$$|u_y/u_x| < C_\varepsilon W . \tag{44}$$

As before, we write

$$u(x, y) = \psi(x)e(x, y) + v(x, y) .$$

First, we estimate $|\nabla v|$. From the analogue of equation (34) without the factor ρ and from Lemma 5 we have

$$|v|, |\Delta v| \leqq CW/L$$

on Ω_0, and this implies

$$|\nabla v| \leqq CW/L \tag{45}$$

near Λ, see (48) in Section 7.

In order to estimate the derivatives of $\bar{u}$, we first remark that the first part of the theorem, together with (26), implies that

$$|\psi(\bar{x}, \bar{y})| \leqq CW/L .$$

From this and (18), we see that $|e_x\psi| < CW^2/L$, and therefore the conclusion of Lemma 2 implies that at $(\bar{x}, \bar{y})$,

$$\tilde{u}_x = e_x\psi + e\psi_x \geqq c\varepsilon/L$$

for large L. Finally,

$$\tilde{u}_y = e_y\psi = O(W/L) ,$$

and together with (45) this implies (44) and finishes the proof of the theorem.

7 Proof of Theorem 4

We begin the proof of Theorem 4 by choosing the point x_2, $a < x_2 < b$ equal to the zero of the second eigenfunction for the ordinary differential operator on $[a, b]$

$$-\frac{d^2}{dx^2} + \frac{\pi^2}{h(x)^2} \tag{46}$$

with Dirichlet boundary conditions. Choose $A = x_2 - L/40$ and $B = x_2 + L/40$. Theorem B of [J] says that if $(x_0, y_0) \in \Lambda$, then $|x_0 - x_2| < K$ where K is an absolute constant. In particular, (6) is satisfied.

In Section 8 we will prove the following lemma.

Lemma 9 *If* $|x - x_2| < L/10$ *then*

$$h(x) \geqq 1 - L^{-2} .$$

This clearly implies the estimate (7).

To prove the bound (8), assume, without loss of generality, that $\max u_+ = 1$ and $\varepsilon = u(B, 1/2) > 0$. Consider the function

$$v(x, y) = \sin(5\pi(x - x_2 + K)/L)\sin(\pi y)$$

which has eigenvalue $\mu = \pi^2(1 + 25/L^2) > \lambda_2$. Note also that $v(B, 1/2)$ is comparable to 1 because $5\pi(B - x_2 + K)/L$ is bounded between absolute constants strictly between 0 and π. Furthermore, $ue^{\sqrt{\lambda_2}t}$ and $ve^{\sqrt{\mu}t}$ are harmonic functions of three variables. Both functions are positive in Ω within a unit distance from $(B, 1/2)$. Carleson's lemma (see [J] Proposition 3.4) implies that $u(x, y) \leqq C\varepsilon$ for $|x - B| < 1$, and then the generalized maximum principle, applied with a barrier that is an explicit radial eigenfunction vanishing on a circle that touches Ω from the outside, gives decay at the boundary. Thus we obtain $u(B, y) \leqq C\varepsilon v(B, y)$ for all y. (This fact can be proved by a simpler direct argument, but we have prepared the way for a similar argument in which v is a less explicit eigenfunction.) The generalized maximum principle (see [PW] and [J]) says that if $\Delta v = -\mu v$ and $\Delta u = -\lambda u$ on a domain $D, v > 0$ on $\bar{D}$, and $\mu \geqq \lambda$, then u/v attains its maximum on the boundary,

$$\max_D u/v = \max_{\partial D} u/v.$$

We can apply the generalized maximum principle in the region $D = \{(x, y) \in \Omega : x_2 - K < x < B\}$. For a direct application of the generalized maximum principle as stated, one needs strict positivity for v, but one can widen slightly the rectangle on which we have defined v and then take a limit so that the strict positivity is not necessary. The generalized maximum principle implies $u(x, y) \leqq C\varepsilon v(x, y)$, for all $x < B$. In particular, $\max_D u_+ \leqq C\varepsilon$.

Next, we wish to carry out a similar barrier argument with a function u_1 which is the first eigenfunction for the region $\Omega_1 = \{(x, y) \in \Omega : x > B - L/50\}$. Let γ be the eigenvalue for u_1. Because Ω_1 is a subset of the set where $u > 0, \gamma > \lambda_2$. Therefore we can also apply the generalized maximum principle to u and u_1. Normalize u_1 so that $\max u_1 = 1$. Proposition A of [J] and Lemma 9 imply that there is an absolute constant $\delta > 0$ such that $u_1(B, 1/2) \geqq \delta$. Using Carleson's lemma the same argument as for v above shows that $u(B, y) \leqq C\varepsilon u_1(B, y)$ for all y and the generalized maximum principle on the set of points (x, y) in Ω such that $x > B$ implies $u(x, y) \leqq C\varepsilon u_1(x, y)$ for all $x > B$.

Combining the estimates of the two preceding paragraphs, we have $1 = \max u_+ \leqq C\varepsilon$. Since $u(A, 1/2) \leqq 0$, we have $u(B, 1/2) - u(A, 1/2) \geqq c_1$, as desired. Let us remark that it is also possible to prove, using the techniques of [J], that $u(B, 1/2) \geqq c_1$ and $u(A, 1/2) \leqq -c_1$ hold simultaneously. In other words the values $\max u_+$ and $\max u_-$ are comparable. This requires a more complicated argument using the fact that on a unit scale, the control of the maximum crosses the boundary $\max_{R_1} u_+ \leqq C \max_{R_1} u_-$ for the region R_1 of points of

Ω within a unit distance of Λ. But this is not needed here and in fact follows from our estimates in the main part of the paper. See especially (26).

Finally, we prove (9). The particular form of v implies

$$u_+(x,y) \leqq C(\max{}_{u_+})(1+|x-x_2|)/L \quad \text{for all } x < B\,. \tag{47}$$

In the remaining portions of Ω, the same inequality holds trivially since $|x-x_2|/L$ is larger than an absolute, positive constant. Furthermore we can replace x_2 by x_0 since their difference is bounded by a constant. This estimate and the similar bound for u_- imply the part of (9) involving just u. The full estimate (9) now follows from the well-known estimate

$$\max_{\Omega_1} |\nabla w| \leqq C \max_{\Omega_2}(|w| + |\Delta w|) \tag{48}$$

where $\Omega_1 = \{(x,y) \in \Omega : c-1 < x < c+1\}$ and $\Omega_2 = \{(x,y) \in \Omega : c-2 < x < c+2\}$, and w is a function vanishing on the part of the boundary of Ω_2 that lies in some neighborhood $\mathcal{N}$ of Ω_1. This follows from the estimates on the Green's function in the convex domain Ω_2 stated in Lemma 7: One writes

$$\Delta(\rho w) = (\Delta\rho)w + 2\nabla\rho\nabla w + \rho\Delta w$$

where ρ is a smooth cut-off function that is one in Ω_1 and zero outside $\mathcal{N}$. Integrating against the Green's function, integrating by parts in the mixed term and then applying ∇ one obtains (48).

8 The lower bound for $h(x)$ near the nodal line

In this section we prove Lemma 9. It should be noted that for our proof of the main theorem the bound $h(x) \geqq 1 - CL^{-2}$ for $|x-x_2| < L/10$ would suffice, with any constant C (only in that case the interval $[A,B]$ would have to be chosen shorter in order for the argument in Section 3 to work). To show that $C=1$ works is barely harder, and the fact is aesthetically pleasing. It involves keeping track precisely of some numerical constants, though.

Let $[\alpha,\beta]$ be the interval used in the definition of L, i.e.

$$\beta - \alpha = L$$

and $h(x) \geqq 1 - L^{-2}$ on $[\alpha,\beta]$.

By the considerations in the Introduction (see the definition of x_1 there), Lemma 9 follows from the following lemma.

Lemma 10 *Let* $\rho = 1/9$ *and set* $\bar{x} = \alpha + \rho L$. *Let*

$$\Omega_- = \Omega \cap \{x < \bar{x}\}, \qquad \Omega_+ = \Omega \cap \{x > \bar{x}\}\,.$$

Then, for sufficiently large L,

$$\lambda_1(\Omega_-) > \lambda_1(\Omega_+)\,.$$

In the proof we will need precise estimates on the first eigenvalue of a long, thin triangle. Given a slope $s \in [0,1]$ and a height $H \in [1/2, 2]$, let $D_{s,H}$ be the isosceles triangle

$$D_{s,H} = \{(x,y) : |y| < sx/2; 0 < x < H/s\} .$$

Recall that the first positive zero of the Bessel function J_v has the asymptotic expansion

$$v + c_1 v^{1/3} + O(v^{-2/3}), \quad c_1 \approx 1.85 \tag{49}$$

for large v.

Lemma 11 *The first Dirichlet eigenvalue satisfies*

$$\lambda_1(D_{s,H}) = \frac{\pi^2}{H^2}(1 + c_0 s^{2/3} + O(s^{4/3}))$$

with $c_0 = 2c_1\pi^{-2/3} \approx 1.72$.

Proof. Define the circular sector

$$\Gamma_{s,R} = \left\{(\theta, r) : |\theta| < \tan^{-1}\frac{s}{2}; 0 < r < R\right\}$$

in polar coordinates. We have

$$\Gamma_{s,H/s} \subset D_{s,H} \subset \Gamma_{s,H\sqrt{1+s^2}/s} .$$

The first eigenfunction for $\Gamma_{s,R}$ is

$$J_v(\sqrt{\lambda_1}r)\cos v\theta$$

where $v = \pi/\tan^{-1}(s/2)$, λ_1 is the eigenvalue and $\sqrt{\lambda_1}R$ equals the first positive zero of J_v. Using (49) one gets the result after a straightforward calculation. □

Proof. (of Lemma 10) The lemma will follow if we can construct domains Ω'_-, Ω'_+ with

$$\Omega'_- \supset \Omega_-, \qquad \Omega^1_+ \subset \Omega_+$$

and prove $\lambda_1(\Omega'_-) > \lambda_1(\Omega'_+)$ by direct calculation.

We begin with Ω'_+: By assumption, Ω_+ is convex and $h(x) \geqq 1 - L^{-2}$ on $[\bar{x}, \beta]$. Therefore, the parallelogram with corners

$$(\bar{x}, f_1(\bar{x})), \quad (\bar{x}, f_1(\bar{x}) + 1 - L^{-2}), \quad (\beta, f_1(\beta)), \quad (\beta, f_1(\beta) + 1 - L^{-2})$$

is contained in Ω_+. Since the slopes of the upper and lower sides are at most $2L^{-2}$, this parallelogram contains a rectangle Ω'_+ with sides

$$l_1 = 1 - L^{-2} - O(L^{-4}), \quad l_2 = \beta - \bar{x} - O(L^{-2}) = (1-\rho)L - O(L^{-2})$$

and so

$$\lambda_1(\Omega'_+) = \pi^2(l_1^{-2} + l_2^{-2}) \tag{50}$$

$$= \pi^2\left[1 + \left(2 + \frac{1}{(1-\rho)^2}\right)L^{-2} + O(L^{-4})\right] . \tag{51}$$

We now construct Ω'_-. If α is the left endpoint a of Ω then we take for Ω'_- the rectangle $[\alpha, \bar{x}] \times [0, 1]$, which has eigenvalue $\pi^2[1 + \rho^{-2}L^{-2}] > \lambda_1(\Omega'_+)$.

If α is not the left endpoint of Ω then Ω'_- will be a triangle. To simplify notation, we will assume that Ω_- is symmetric with respect to a horizontal axis, which we choose as x-axis; that is,

$$\Omega_- = \{(x, y) : a < x < \bar{x}; |y| < h(x)/2\} .$$

This is actually no loss of generality since the first eigenvalue decreases upon symmetrization (see [S]).

Denote

$$s = h'(\alpha^-), \qquad S = L^3 s .$$

Since $\max_{[\alpha,\beta]} h \geqq 1 - 12L^{-3}$ by Lemma 2.1 in [J] and $h(\alpha) = 1 - L^{-2}$ by definition of L, convexity of h implies

$$S \geqq L^3(L^{-2} - 12L^{-3})/L = 1 - O(L^{-1}) . \tag{52}$$

Fix a constant S_0, to be determined below. We will distinguish two cases, according to whether the slope s is small or big. We will take for Ω'_- an isosceles triangle formed by the vertical line $x = \bar{x}$ as base and

(Case $S < S_0$) the tangents to Ω of slopes $\pm s/2$ through $(\alpha, \pm h(\alpha)/2)$.

(Case $S \geqq S_0$) the lines of slopes $\pm S_0 L^{-3}/2$ through $(\alpha, \pm 1/2)$.

Clearly, $\Omega'_- \supset \Omega_-$ in both cases. The construction of the first case does not work for $S \geqq S_0$ since the length of the base is too big then, which makes the first eigenvalue of the triangle too small, by Lemma 11.

In the case $S < S_0$, Ω'_- is congruent to the triangle $D_{s,H}$ with

$$H = h(\alpha) + s(\bar{x} - \alpha) = 1 - L^{-2} + S\rho L^{-2} .$$

From Lemma 11 we then get

$$\begin{aligned}\lambda_1(\Omega'_-) &= \pi^2[1 + 2L^{-2} - 2S\rho L^{-2} + c_0 S^{2/3} L^{-2} + O(L^{-4})] \\ &= \pi^2[1 + (2 + S^{2/3}(c_0 - 2\rho S^{1/3}))L^{-2} + O(L^{-4})] ,\end{aligned}$$

and from (52) and (51) we see that, for large $L, \lambda_1(\Omega'_-) > \lambda_1(\Omega'_+)$ is satisfied if

$$2 + c_0 - 2\rho S_0^{1/3} > 2 + (1 - \rho)^{-2} . \tag{53}$$

In the case $S \geqq S_0, \Omega'_-$ is congruent to the triangle $D_{S_0 L^{-3}, H}$ with

$$H = 1 + S_0 \rho L^{-2} .$$

From Lemma 11 we get

$$\lambda_1(\Omega'_-) = \pi^2[1 + L^{-2} S_0^{2/3}(c_0 - 2\rho S_0^{1/3}) + O(L^{-4})] ,$$

so we have $\lambda_1(\Omega'_-) > \lambda_1(\Omega'_+)$ if

$$S_0^{2/3}(c_0 - 2\rho S_0^{1/3}) > 2 + (1-\rho)^{-2} . \tag{54}$$

It remains to choose S_0 and ρ so that (53, 54) are satisfied. This is possible precisely because

$$c_0 > 1 .$$

One just needs to choose $2\rho S_0^{1/3}$ smaller than $c_0 - 1$, and then take ρ sufficiently small and S_0 sufficiently large. For example, one can take $2\rho S_0^{1/3} = c_0/4$ and then $\rho = 1/9$. □

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