Note

Counting Complements in the Partition Lattice, and Hypertrees

DANIEL GRIESER*

Department of Mathematics, Massachusetts Institute of Technology,
Cambridge, Massachusetts 02139

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Given two partitions \( \pi, \sigma \) of the set \( [n] = \{1, \ldots, n\} \) we call \( \pi \) and \( \sigma \) complements if their only common refinement is the partition \( \{\{1\}, \ldots, \{n\}\} \) and the only partition refined by both \( \pi \) and \( \sigma \) is \( [n] \). If \( \pi = \{A_1, \ldots, A_m\} \) then we write \( |\pi| = m \). We prove that the number of complements \( \sigma \) of \( \pi \) satisfying \( |\sigma| = n - m + 1 \) is

\[
\prod_{i=1}^{m} |A_i| \cdot (n - m + 1)^{m-2}.
\]

For the proof we assign to each \( \sigma \) a hypertree describing the pattern of intersections of blocks of \( \pi \) and \( \sigma \) and then count the number of hypertrees and the number of \( \sigma \) corresponding to each hypertree. © 1991 Academic Press, Inc.

1. Problem and Sketch of Solution

A partition \( \pi = \{A_1, \ldots, A_m\} \) of the set \( [n] = \{1, \ldots, n\} \) is an (unordered) family of nonempty subsets \( A_1, \ldots, A_m \) of \( [n] \) which are pairwise disjoint and whose union is \( [n] \). We call the \( A_i \) the blocks of \( \pi \), and let \( |\pi| = m \). A partition \( \{B_1, \ldots, B_r\} \) is a refinement of \( \{A_1, \ldots, A_m\} \) if each \( B_j \) lies in some \( A_i \). It is well known (but of no relevance in this paper) that the ordering relation so defined on the set of all partitions of \( [n] \) makes it into a lattice. Two partitions \( \pi \) and \( \sigma \) of \( [n] \) are complements if their only common refinement is \( \{\{1\}, \ldots, \{n\}\} \) (we then write \( \pi \wedge \sigma = \emptyset \)) and the only partition refined by both \( \pi \) and \( \sigma \) is \( [n] \) (we then write \( \pi \vee \sigma = [n] \)). We will prove:

* Current address: Department of Mathematics, University of California at Los Angeles, Los Angeles, California 90024.
THEOREM 1. If $\pi = \{A_1, ..., A_m\}$ is a partition of $[n]$, then the number of complements $\sigma$ of $\pi$ with $|\sigma| = n - m + 1$ is

$$\prod_{i=1}^{m} |A_i| \cdot (n - m + 1)^{m - 2}.$$  \hspace{2cm} (1)

Throughout the paper, $\pi = \{A_1, ..., A_m\}$ will be a fixed partition of $[n]$, and we denote by $\Sigma$ the set of partitions $\sigma$ counted in the theorem. In order to prove (1) we will split up $\Sigma$ into pieces and count the pieces and their cardinalities. We will assign to each $\sigma \in \Sigma$ a hypertree $H_\sigma$, and one piece will be the set of $\sigma$ belonging to one fixed hypertree $H$. It will turn out that the cardinality of the piece depends only and in a simple manner on the vertex degrees (the degree sequence) of $H$. We will give a formula for the number of hypertrees with a fixed degree sequence, and summing over all degree sequences will yield (1).

The relevant definitions and facts about hypertrees are given in Section 2, and the proof of Theorem 1 in Section 3. Some remarks as to possible simplifications and generalizations conclude the paper.

2. Hypertrees

A hypergraph $H = (V, E)$ consists of a finite vertex set $V$ and a finite family $E$ of nonempty subsets of $V$, the set of edges. $E$ may contain multiple elements (multiple edges). A one-element edge is a singleton or loop. The degree of vertex $v$ is the number of edges containing $v$. A path from vertex $v$ to vertex $w$ is a sequence $v = v_0, e_1, v_1, ..., e_r, v_r = w$ of distinct vertices $v_0, ..., v_r$ and distinct edges $e_1, ..., e_r$ such that each $e_i$ contains its two neighbors. If $v = w$ but otherwise all vertices are distinct, and if $r \geq 2$, we speak of a cycle.

$H$ is connected if there is a path between any two vertices, or, equivalently, if for any proper subset $V'$ of $V$ there is an edge intersecting both $V'$ and $V - V'$. $H$ is a hypertree if $H$ is connected and contains neither loops nor cycles. In particular, a hypertree has no multiple edges.

The well-known fact that a connected graph on $m$ vertices has at least $m - 1$ edges, and exactly $m - 1$ edges if and only if it is a tree, generalizes easily to:

LEMMMA 1. Let the hypergraph $H = (V, E)$ be connected and loop free. Then

$$\sum_{e \in E} |e| \geq |V| + |E| - 1,$$ \hspace{2cm} (2)

and equality holds if and only if $H$ is a hypertree.
Proof. We replace each edge $e = \{v_1, \ldots, v_{|e|}\}$ by $|e| - 1$ edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{|e|-1}, v_{|e|}\}$ (numbering arbitrary). Obviously, the resulting graph is connected if and only if $H$ is, and is a tree if and only if $H$ is a hypertree. Now (2) is equivalent to

$$\sum_{e \in E} (|e| - 1) \geq |V| - 1,$$

and the assertion follows from the analogue for graphs. \]

We remark that if in the hypergraph $H = (V, E)$ on the vertex set $V = [m]$ the vertices $1, \ldots, m$ have degrees $d_1, \ldots, d_m$, respectively, then $\sum_{e \in E} |e| = \sum_{i=1}^m d_i$.

The following formula appeared in [2], but for completeness we give a proof here.

**Lemma 2.** Let $m, k, d_1, \ldots, d_m$ be positive integers, $\sum_{i=1}^m d_i = m + k - 1$. Then the number of hypertrees on the vertex set $[m]$ with $k$ edges in which vertex $i$ has degree $d_i$ $(i = 1, \ldots, m)$ is

$$h(m, k; d_1, \ldots, d_m) = S_{m-1, k} \left( \begin{array}{c} k-1 \\ d_1 - 1 \ldots d_m - 1 \end{array} \right),$$

where $S_{m-1, k}$ is the Stirling number of the second kind.

**Proof.** By induction on $m$. For $m = 1$ or $k = 1$ the claim is true. Let $m > 1, k > 1$.

If $k \geq m$ then $\sum_{e \in E} |e| = \sum_{i=1}^m d_i \leq 2k - 1$, hence some edge must be a loop, and there is no hypertree. Hence assume $k \leq m - 1$.

Then some $d_i$ must be one. Let $d_1 = 1$, and let $H$ be a hypertree with degrees $d_1, \ldots, d_m$, and let $e$ be the edge containing vertex 1.

Assume first $|e| = 2$, $e = \{1, i\}$. Let $H'$ be the hypergraph obtained by removing vertex 1 and edge $e$ from $H$. $H'$ is a hypertree on $m - 1$ vertices, with $k - 1$ edges and vertex degrees $d_2, \ldots, d_i - 1, \ldots, d_m$, and $d_i - 1 > 0$ because $k > 1$ and $H'$ is connected. Hence there are $h(m - 1, k - 1; d_2, \ldots, d_i - 1, \ldots, d_m)$ possible $H'$'s, and because we can recover $H$ from $i$ and $H'$ the number of $H$'s with $|e| = 2$ is

$$\sum_{i=2}^m h(m - 1, k - 1; d_2, \ldots, d_i - 1, \ldots, d_m) = S_{m-2, k-1} \left( \begin{array}{c} k-1 \\ d_2 - 1 \ldots d_m - 1 \end{array} \right).$$
If $|e| > 2$ then we remove only vertex 1 and obtain a hypertree $H'$ on $m - 1$ vertices, with $k$ edges and vertex degrees $d_2, \ldots, d_m$. We can recover $H$ from the knowledge of $H'$ and of the edge which contained vertex 1. Hence the number of $H$'s with $|e| > 2$ is

$$kh(m - 1, k; d_2, \ldots, d_m) = kS_{m-2,k} \left( \frac{k-1}{d_2 - 1 \cdots d_m - 1} \right).$$

Now the recursion $S_{m-1,k} = S_{m-2,k-1} + kS_{m-2,k}$ yields the result. \qed

Although we will not need it here, we note the immediate

**COROLLARY 1.** The number of hypertrees with $m$ vertices and $k$ edges is $S_{m-1,k} m^{k-1}$.

A hypertree is essentially the same as a graph all of whose blocks are complete graphs. Viewed in this way, the corollary follows also from the well-known block-tree-theorem which gives a formula for the number of graphs with prescribed blocks (see [13]).

3. **Solution**

Let $\pi = \{A_1, \ldots, A_m\}$ be fixed and $\sigma = \{B_1, \ldots, B_r\}$ be any partition of $[n]$, with

$$|B_1| \geq |B_2| \geq \cdots \geq |B_k| > 1 = |B_{k+1}| = \cdots = |B_r|.$$

Let $H_\sigma$ be the hypergraph with vertex set $[m]$ and edges $C_1, \ldots, C_k$, where

$$C_j = \{i \in [m] \mid A_i \cap B_j \neq \emptyset\} \quad (j = 1, \ldots, k).$$

Thus we think of the blocks of $\pi$ as vertices, and a set of vertices is an edge if for some $j \in [k]$, $B_j$ has elements in common with exactly these vertices. Hence the number of edges of $H_\sigma$ is just the number of nonsingleton blocks of $\sigma$.

**LEMMA 3.** $H_\sigma$ has the following properties:

(i) $|C_j| \leq |B_j|$ for $j = 1, \ldots, k$, and equality holds for all $j$ iff $\pi \wedge \sigma = \emptyset$.

(ii) $H_\sigma$ is connected iff $\pi \vee \sigma = \hat{1}$. 
(iii) If \( \pi \vee \sigma = \emptyset \), then \( |\sigma| \leq n + 1 - |\pi| \), and equality holds iff \( \pi \cap \sigma = \emptyset \) and \( H_\sigma \) is a hypertree.

Proof. (i) \( |C_j| \leq |B_j| \) is clear, and \((\pi \wedge \sigma = \emptyset \iff A_i \cap B_j| \leq 1, \forall i \in [m], j \in [k])\) implies the second part.

(ii) \( \pi \vee \sigma = \emptyset \iff \) there are no proper subsets \( I \subseteq [m] \) and \( J \subseteq [r] \) for which \( \bigcup_{i \in I} A_i = \bigcup_{j \in J} B_j \iff \) for all proper subsets \( I \subseteq [m] \) there is a \( C_j \) intersecting both \( I \) and \([m] - I \iff H_\sigma \) is connected.

(iii) If \( \pi \vee \sigma = \emptyset \) then \( H_\sigma \) is connected, hence \( \sum_{j=1}^k |C_j| \geq m + k - 1 \) by Lemma 1. By (i), \( \sum_{j=1}^k |C_j| \leq \sum_{j=1}^k |B_j| = n - (r - k) \), and we obtain \( r \leq n + 1 - m \), with equality iff \( \sum_{j=1}^k |C_j| = \sum_{j=1}^k |B_j| \) (hence \( |C_j| = |B_j| \) and \( H_\sigma \) has no loops) and \( \sum_{j=1}^k |C_j| = m + k - 1 \) which by Lemma 1 and (i) is equivalent to \( \pi \wedge \sigma = \emptyset \) and \( H_\sigma \) hypertree. \( \blacksquare \)

Now we count the number of complements having the same hypertree.

**Lemma 4.** Given a hypertree \( H \), there are \( \prod_{i=1}^m (a_i)_{d_i} \) complements \( \sigma \) of \( \pi \) with \( H_\sigma = H \). Here \( d_i \) is the degree of vertex \( i \) in \( H \), \( a_i = |A_i| \) and \( (\alpha)_{\beta} = \alpha(\alpha - 1) \cdots (\alpha - \beta + 1) \).

Proof. Let \( C_1, \ldots, C_k \) be the edges of \( H \). We obtain all \( \sigma \) by first picking an element from every \( A_i \) with \( i \in C_1 \), thus composing \( B_1 \), then picking elements for \( B_2 \), etc. In the end we will have picked \( d_i \) elements from \( A_i \) for every \( i \), one after another. This is possible in \( \prod_{i=1}^m (a_i)_{d_i} \) ways, and each way gives a different \( \sigma \) because \( H \) has no multiple edges. \( \blacksquare \)

Proof of Theorem 1. We first count the number of complements \( \sigma \), \( |\sigma| = n - m + 1 \), with a fixed number \( k \) of nonsingleton blocks. By Lemma 3(iii) any complement \( \sigma \) for which \( H_\sigma \) is a hypertree has \( n - m + 1 \) blocks, and by Lemmas 2 and 4 this number equals (with \( p = \prod_{i=1}^m a_i \))

\[
\sum_{\sum_{d_1, \ldots, d_m \geq 1} d_1 + \cdots + d_m = m + k - 1} S_{m-1,k} \left( \begin{array}{c} k-1 \\ d_1-1 \cdots d_m-1 \end{array} \right) \prod_{i=1}^m (a_i)_{d_i} \\
= pS_{m-1,k} \sum_{\sum_{e_1, \ldots, e_m \geq 0} e_1 + \cdots + e_m = k - 1} \left( \begin{array}{c} k-1 \\ e_1 \cdots e_m \end{array} \right) \prod_{i=1}^m (a_i-1)_{e_i} \\
= pS_{m-1,k} \left( \sum_{i=1}^m (a_i-1) \right)_{k-1} \\
= pS_{m-1,k}(n-m)_{k-1}.
\] (3)
Here we used

**Lemma 5.** For nonnegative integers \( r, m, u_1, \ldots, u_m \)

\[
(u_1 + \cdots + u_m) = \sum_{e_1, \ldots, e_m \geq 0} \left( \sum_{e_1 + \cdots + e_m = r} (u_i)_{e_i} \right)
\]

*Proof.* Let \( U_i \ (i = 1, \ldots, m) \) be sets, \( |U_i| = u_i \), and \( U = \bigcup_{i=1}^m U_i \). The left hand side counts the number of ways to pick \( r \) elements in order from \( U \); the right hand side counts the same, first determining which of the \( r \) elements are to be picked from each \( U_i \).

Now summing over \( k \) yields

\[
p \sum_{k=0}^{m-1} S_{m-1,k}(n-m)_{k-1} = p(n-m+1)^{m-2}
\]

by a well-known formula.

### 4. Remarks

The proof rests on the formula for the number of hypertrees which was proved by induction. But it seems that a more direct combinatorial proof, avoiding induction, should be possible, perhaps a generalization of the Prüfer sequence procedure used to enumerate trees (see [3]). Also, formulae (3) and (1) appear to demand a more direct proof.

The more general problem to determine the number \( C(\pi, r) \) of complements of \( \pi \) with \( r \) blocks seems to be much more difficult. By Lemma 3 we have \( C(\pi, r) = 0 \) if \( r > n - m + 1 \). If, for \( r < n - m + 1 \), we want to use the same method, we have to count hypergraphs containing cycles; also, some care would have to be taken because of the possible occurrence of multiple edges (see proof of Lemma 4).

If we drop the condition \( \pi \vee \sigma = \hat{1} \) and require only \( \pi \wedge \sigma = \hat{0} \) and \( |\sigma| = r \) the problem becomes much easier. Using inversion, we obtain for the number of these \( \sigma \)

\[
S(\pi, r) = \frac{1}{r!} \sum_{l=0}^{r} \binom{r}{l} (-1)^{r-l} \prod_{A \in \pi} (l)_{|\sigma|},
\]
see [4], end of Section 3. It is also proved there that the polynomial
\[ \sum_{r \geq 0} S(\pi, r) x^r \] has only real, nonpositive roots. It would be interesting to
settle the corresponding problem for the numbers \( C(\pi, r) \).

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REFERENCES