Negative-weight percolation

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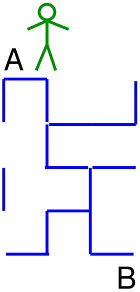
Outline

- Percolation problem
- Algorithms
- Results: two-dimensional, higher dimensions
- Summary

Motivation

- Agent travels:
  "I want to get from A $\rightarrow$ B"

  $\leftrightarrow$ standard (connectivity) percolation



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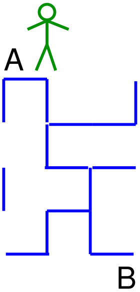
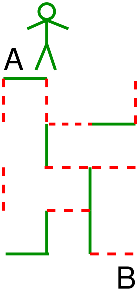


- Agent travels:
  pays for travel resources (positive)
  can earn resources (negative payment)
  "I want to make a profit going from A $\rightarrow$ B"

  $\leftrightarrow$ negative-weight percolation



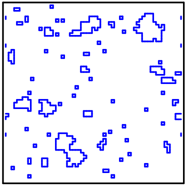
Model

- $L \times L$ lattice, fully periodic boundary conditions
- Undirected edges, weight (cost) distribution:

$$P(\omega) = \rho\,(2\pi)^{-1/2}\exp(-\omega^2/2) \;+\; (1-\rho)\,\delta(\omega-1)$$

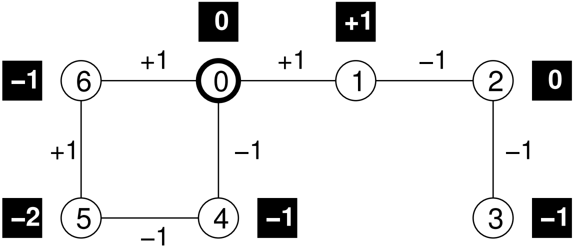
- Allows for loops $\mathcal{L}$ with negative weight $\omega_{\mathcal{L}}$
- Agent on lattice edges: pay/receive resources
- Configuration $\mathcal{C}$ of loops, with

$$E \equiv \sum_{\mathcal{L}\in\mathcal{C}} \omega_{\mathcal{L}} \stackrel{!}{=} \min$$



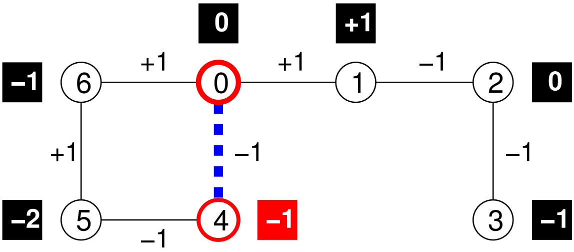
- Obtain $\mathcal{C}$ through mapping to minimum weight perfect matching problem [O. Melchert & AKH, New J. Phys. 2008]

Minimal distances



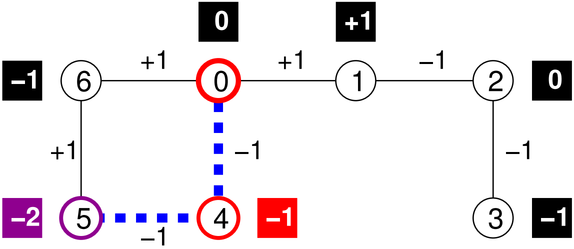
- $d(i) = \min_{j \in N(i)}(d(j) + \omega(i,j))$ not fulfilled

Minimal distances



- $d(i) = \min_{j \in N(i)} (d(j) + \omega(i, j))$ not fulfilled

Minimal distances



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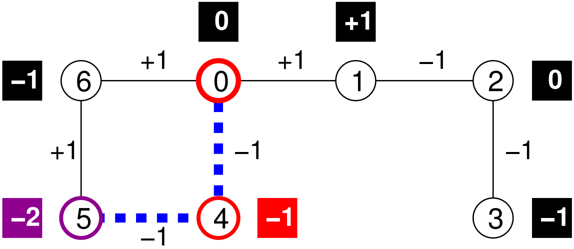

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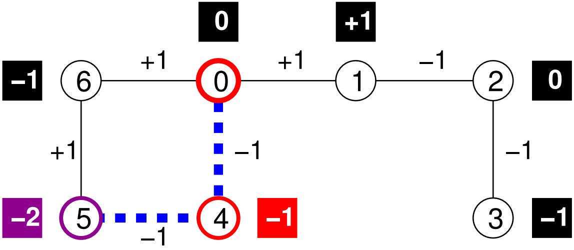
Minimal distances



- $d(i) = \min_{j \in N(i)}(d(j) + \omega(i,j))$ not fulfilled
- Standard minimum-weight path algorithms, e.g. Dijkstra, Bellman-Ford, Floyd-Warshall, don't work
- Minimum-weight path problem requires matching techniques

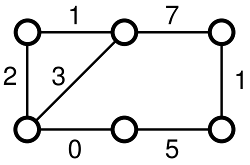
  [R.K. Ahuja, T.L. Magnanti and J.B. Orlin, *Network flows*]

Algorithm – Outline

Brief description of the basic steps:

- Set up auxiliary graph
- Find minimum weight perfect matching (MWPM) on auxiliary graph
- Interpret MWPM as minimal weighted set of paths/loops

- Graph $G = (V, E)$:



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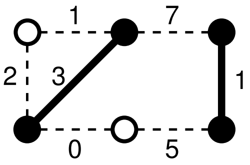
- Matching $M \subset E$:



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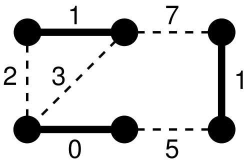
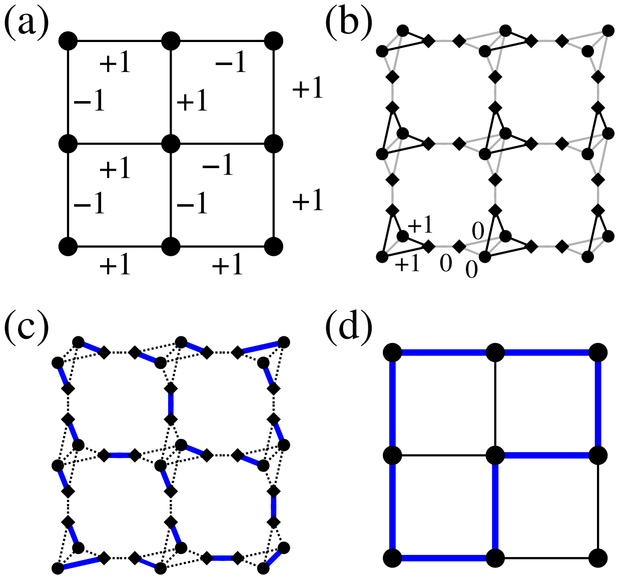
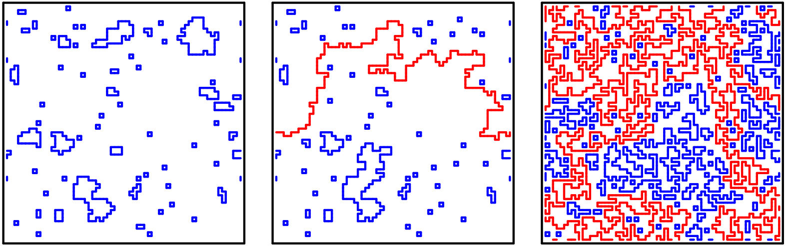
- MWPM, $\omega_M = 2$



Algorithm – Mapping procedure



Loop percolation

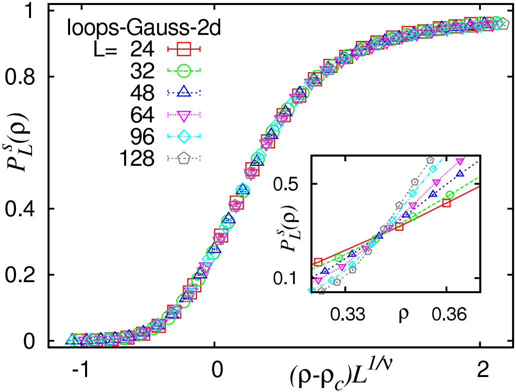


($L = 64$ at $\rho = 0.335,\ 0.340,\ 0.750$)

- Observe system spanning loops above critical ρ
- Disorder induced, geometric transition
- Characterize loops using observables from percolation theory (finite-size scaling (FSS) analysis)

  [D. Stauffer, A. Aharony, *Introduction to Percolation Theory*]

Percolation probability



Percolation probability exhibits FSS:

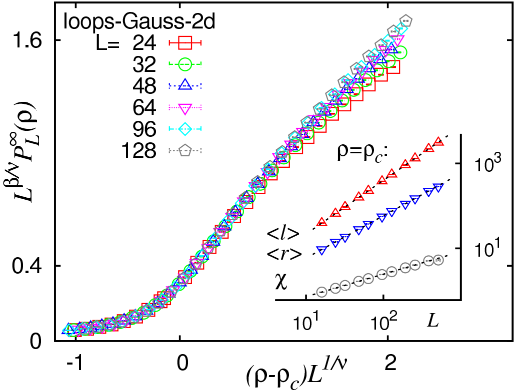
$$P_L^s \sim f[(\rho - \rho_c) L^{1/\nu}]$$

$\rho_c = 0.340(1)$
$\nu = 1.49(7)$
(rand. perc.: $\nu = 1.33$)
$S = 0.91$

- S = "quality" of the scaling assumption
- Similar scaling for mean number of spanning loops
- Compatible results for spanning negative-weight paths.

Percolation strength

- Probability $P_L^\infty \equiv \langle \ell \rangle / L^d$ that edge belongs to percolating loop, finite-size susceptibility $\chi \equiv L^{-d}(\langle \ell^2 \rangle - \langle \ell \rangle^2)$



Exhibits FSS:

$P_L^\infty \sim L^{-\beta/\nu} f[(\rho - \rho_c)L^{1/\nu}]$

$\beta = 1.07(6)$
$S = 1.16$

Percolation strength

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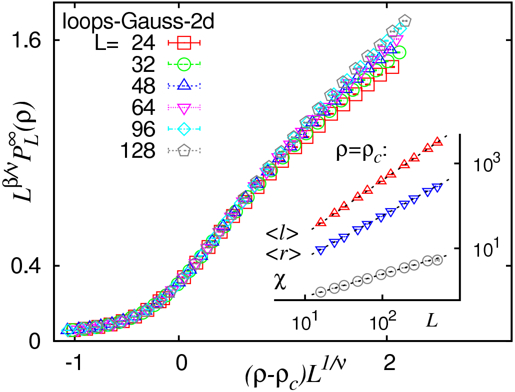


At ρ_c ($L_{max} = 512$):

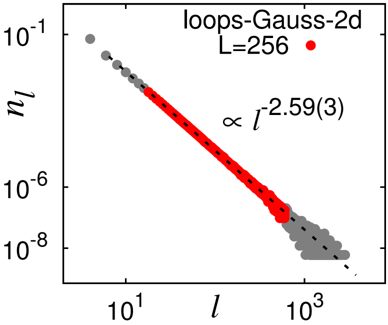
loop length $\langle \ell \rangle \sim L^{d_f}$,
roughness $\langle r \rangle \sim L^{d_r}$,
suscept. $\chi \sim L^{\gamma/\nu}$

$d_f = 1.266(2)$
$d_r = 1.001(4)$
$\gamma = 0.77(7)$

- Scaling relations $d_f = d - \beta/\nu$ and $\gamma + 2\beta = d\nu$ are fulfilled

Fisher exponent

Distribution n_ℓ of the loop lengths ℓ at ρ_c for $L = 256$



Expected FSS:

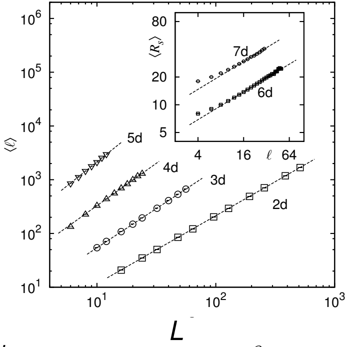
$$n_\ell \sim \ell^{-\tau}$$

$$\tau = 2.59(3)$$

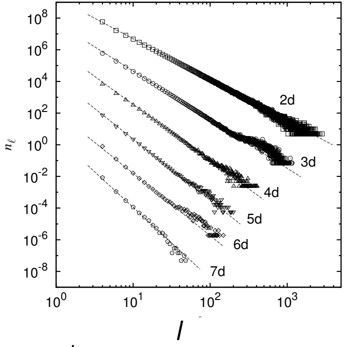
- Excluding spanning loops
- Consistent with scaling relation $\tau = 1 + d/d_f$

High dimensions

Average loop length



Loop length distribution



d	ρ_c	ν	β	γ	d_f	τ
2	0.340(1)	1.49(7)	1.07(6)	0.77(7)	1.266(2)	2.59(3)
3	0.1273(3)	1.00(2)	1.54(5)	-0.09(3)	1.459(3)	3.07(1)
4	0.0640(2)	0.80(3)	1.91(11)	-0.66(5)	1.60(1)	3.55(2)
5	0.0385(2)	0.66(2)	2.10(12)	-1.06(7)	1.75(3)	3.86(3)
6	0.0265(2)	0.50(1)	1.92(6)	-0.99(3)	2.00(1)	4.00(2)
7	0.0198(1)	0.41(1)	–	–	2.08(8)	4.50(1)

O.Melchert, L. Apolo, AKH, arXiv:1003.1591 (2010)

$\rightarrow$ upper critical dimension = 6

Summary

- Negative-weight percolation of loops
- Distinct from random bond/site percolation
- $2d$: critical exponents close to RBIM
- upper critical dimension: 6
- More details:
  L.Apolo, O. Melchert & AKH, Phys. Rev. E 79, 031103 (2009)

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- $2d$: critical exponents close to RBIM
- upper critical dimension: 6
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- Thank you for your attention!

  New book (do better simulations):
  AKH, *Practical Guide to Computer Simulations*,
  World Scientific 2009