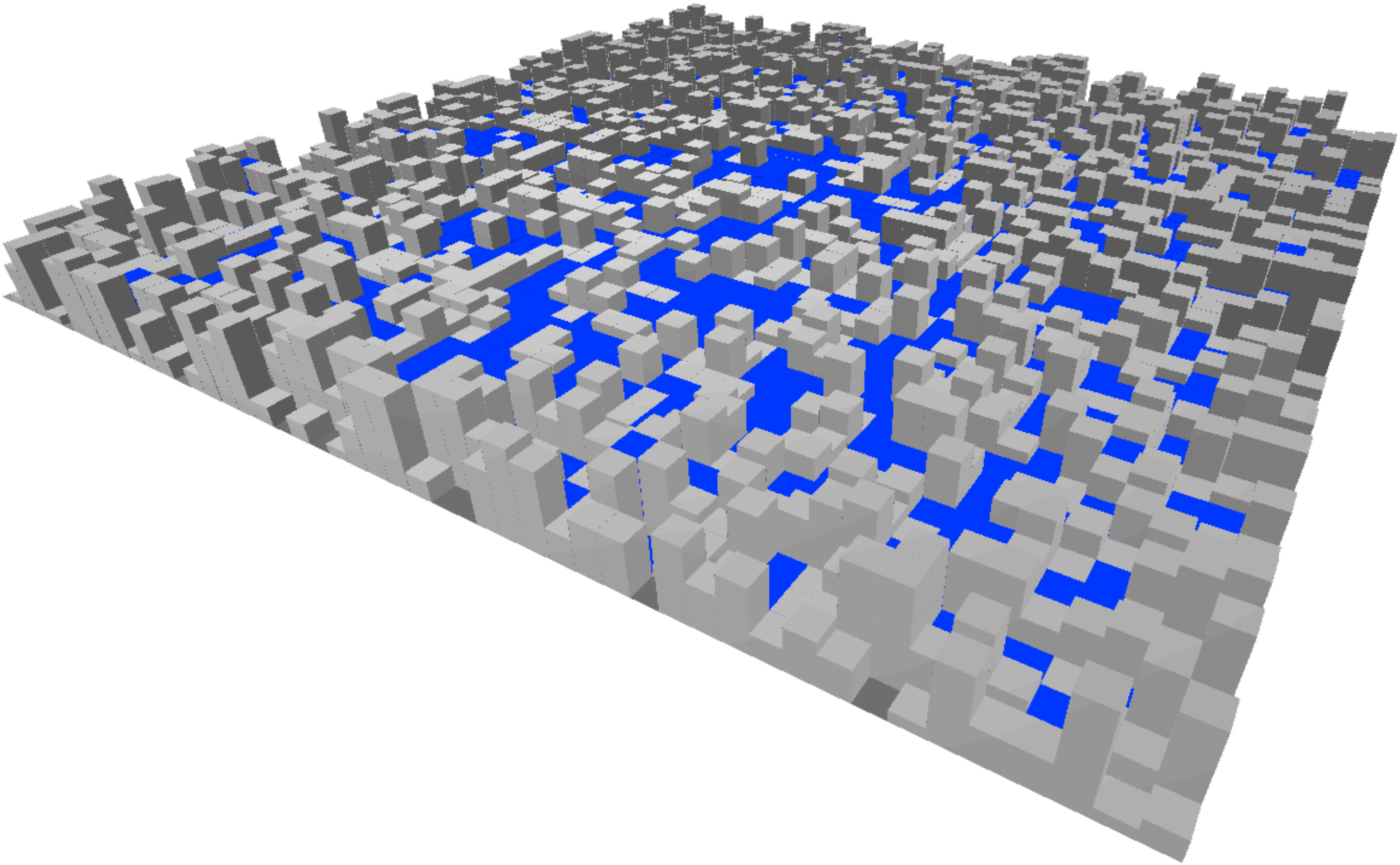
Invasion percolation and the retention of water on random surfaces



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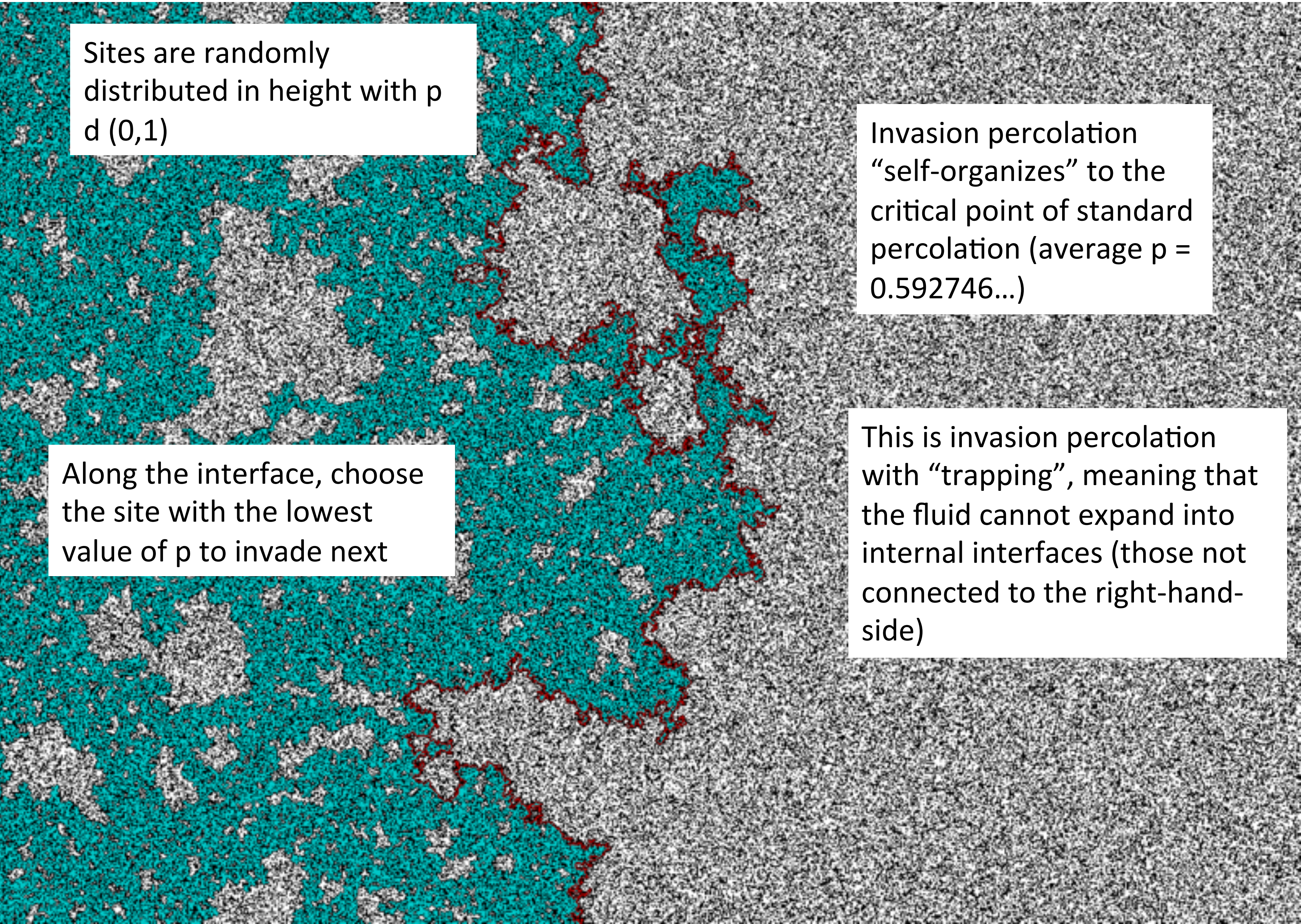
Traditional Invasion Percolation



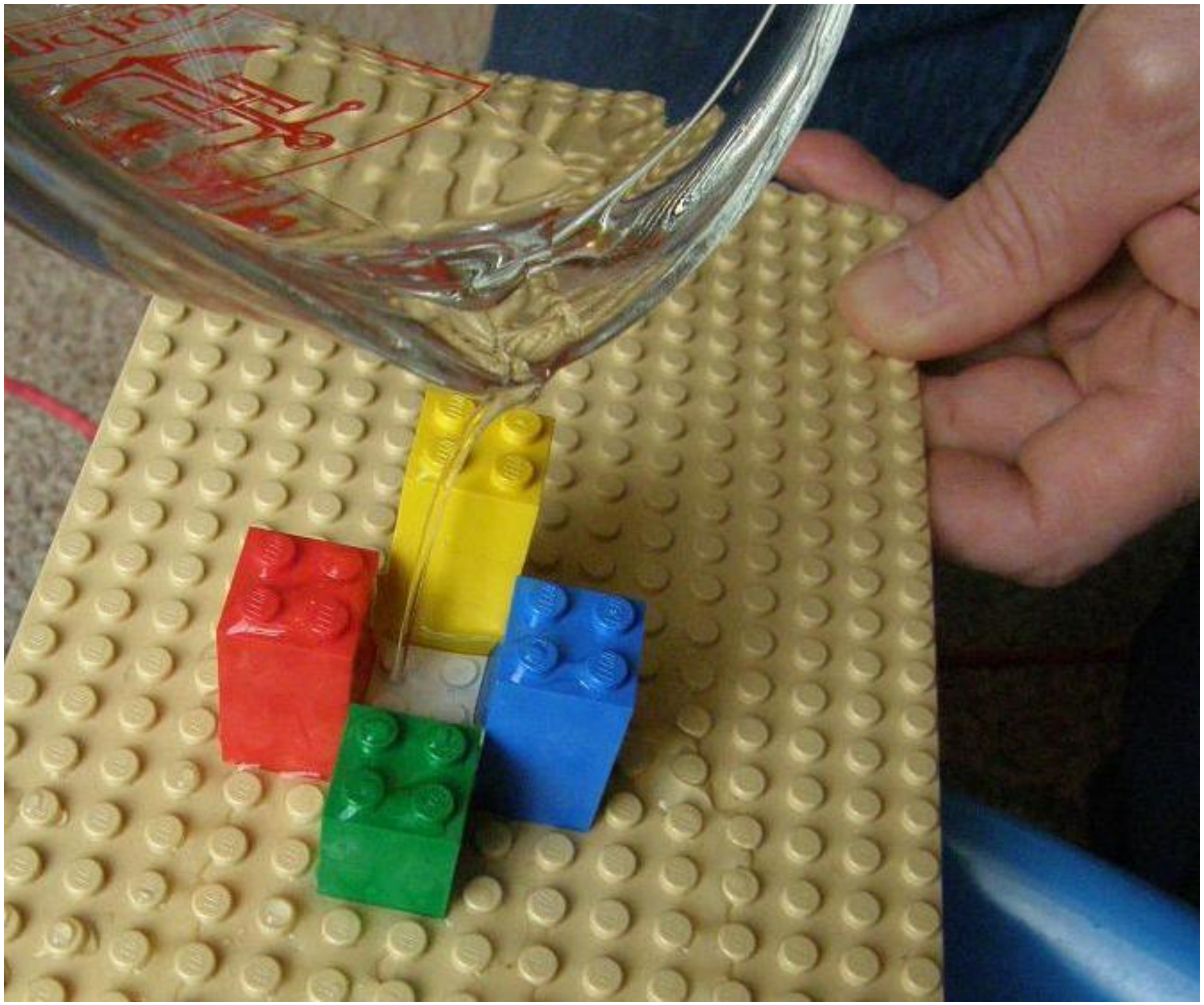
Sites are randomly distributed in height with p d (0,1)

Along the interface, choose the site with the lowest value of p to invade next

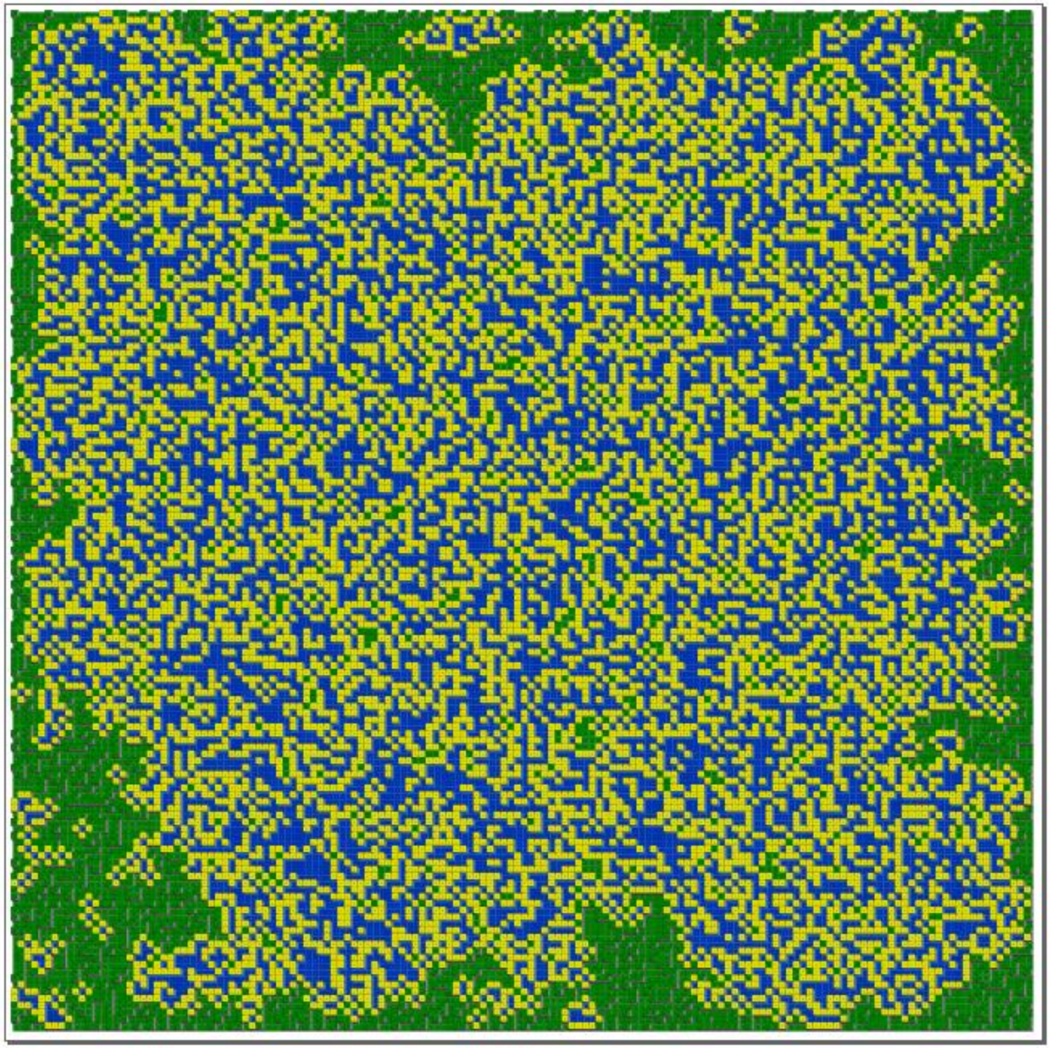
Invasion percolation “self-organizes” to the critical point of standard percolation (average p = 0.592746…)

This is invasion percolation with “trapping”, meaning that the fluid cannot expand into internal interfaces (those not connected to the right-hand-side)

Water retention on a surface – a form of invasion percolation where the water is introduced at an internal site, or at all internal sites

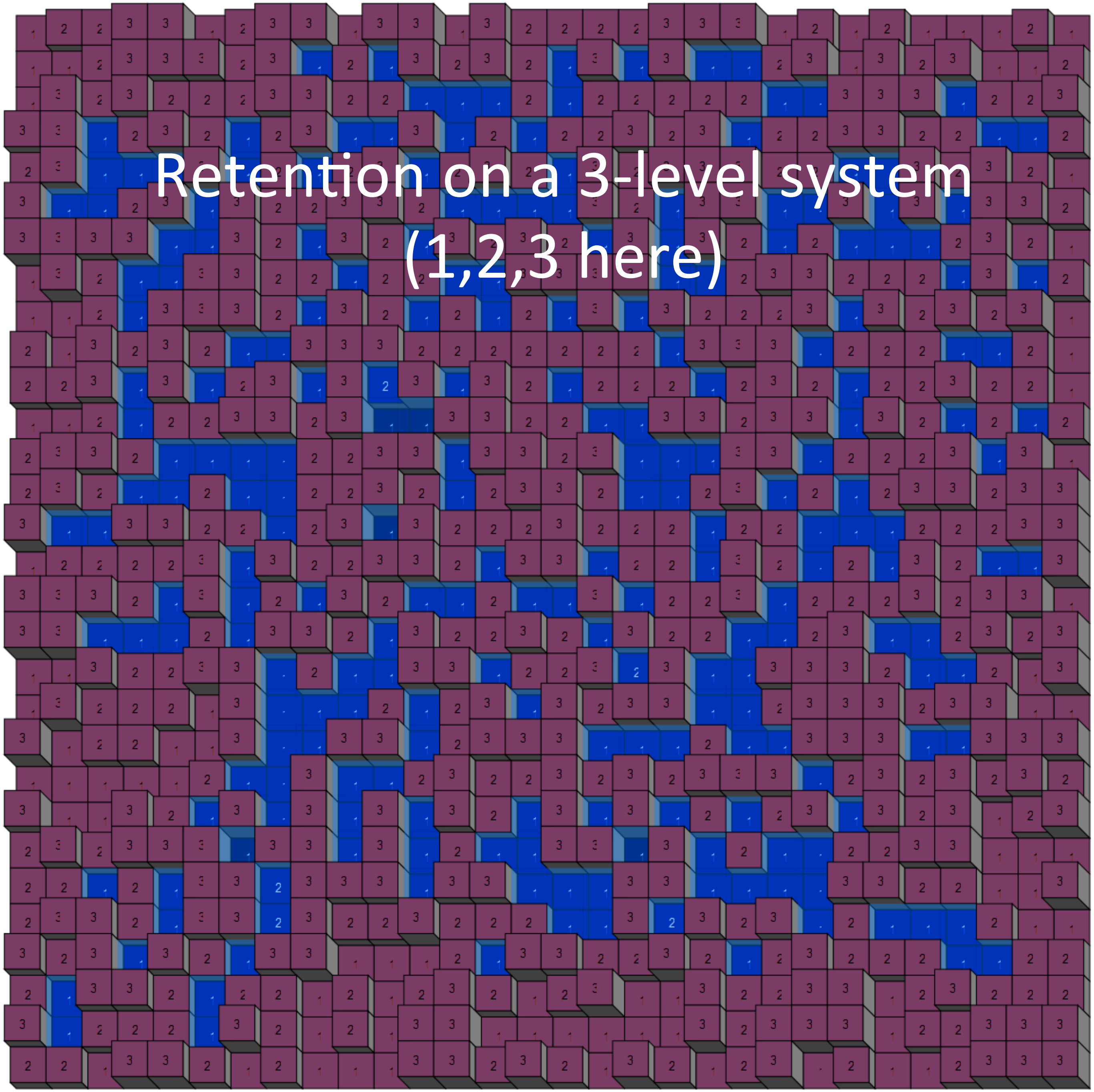


Retention on a discrete 2-level system, equal probability



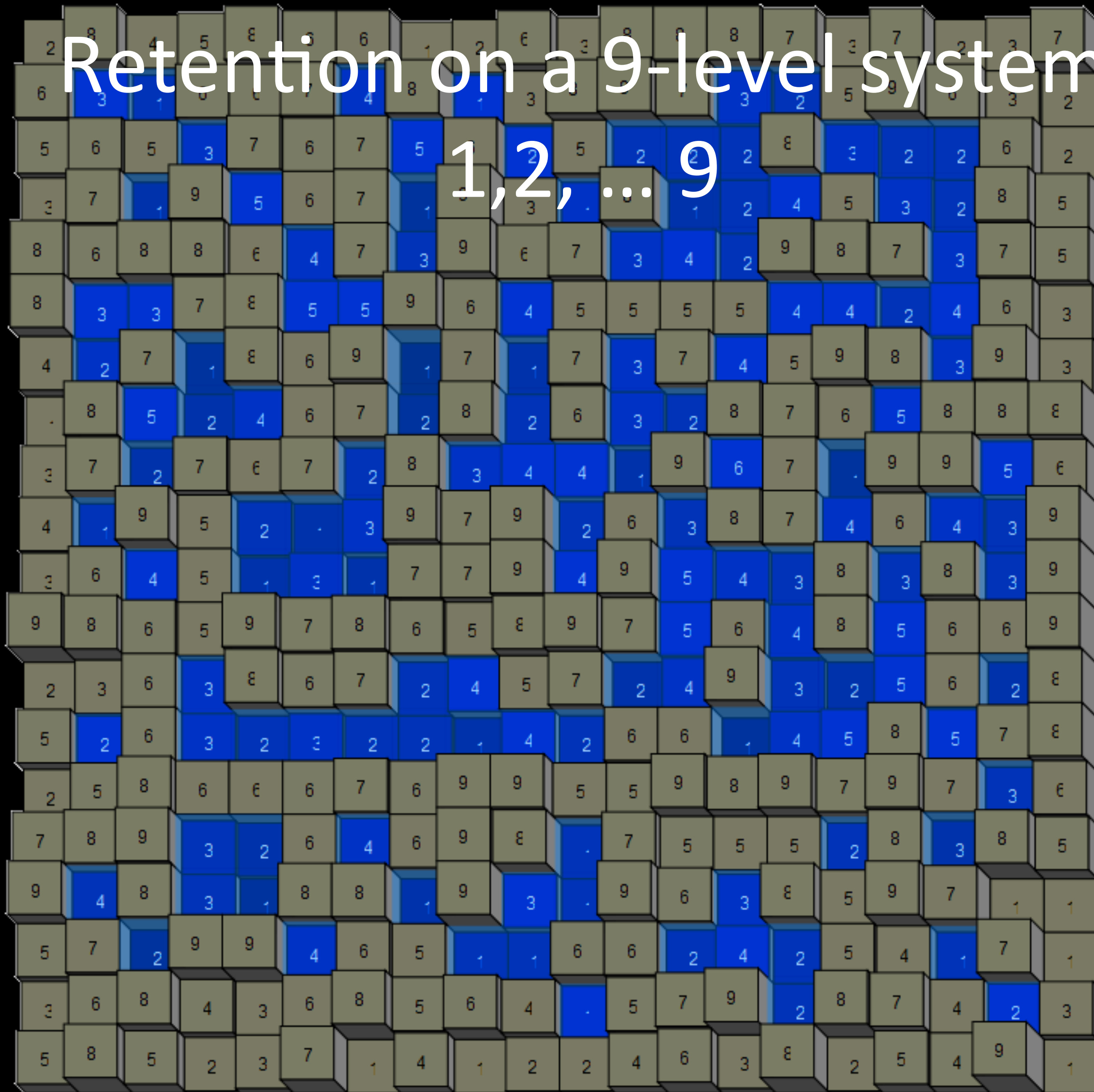
150 x 150 random fill 1,2
9364 units

Retention on a 3-level system
(1,2,3 here)

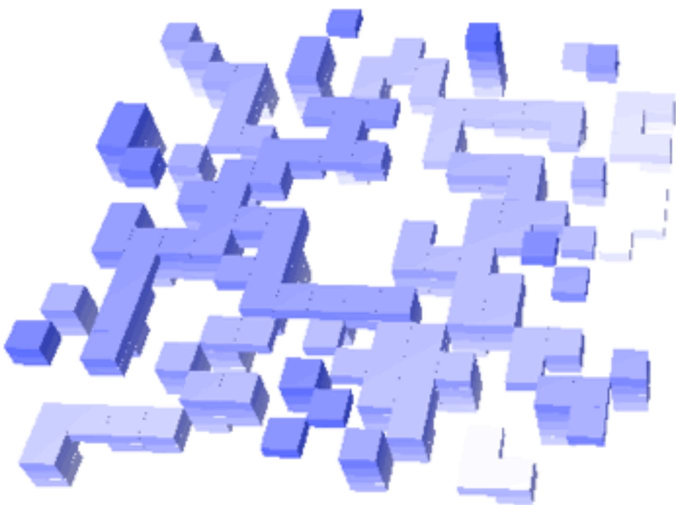


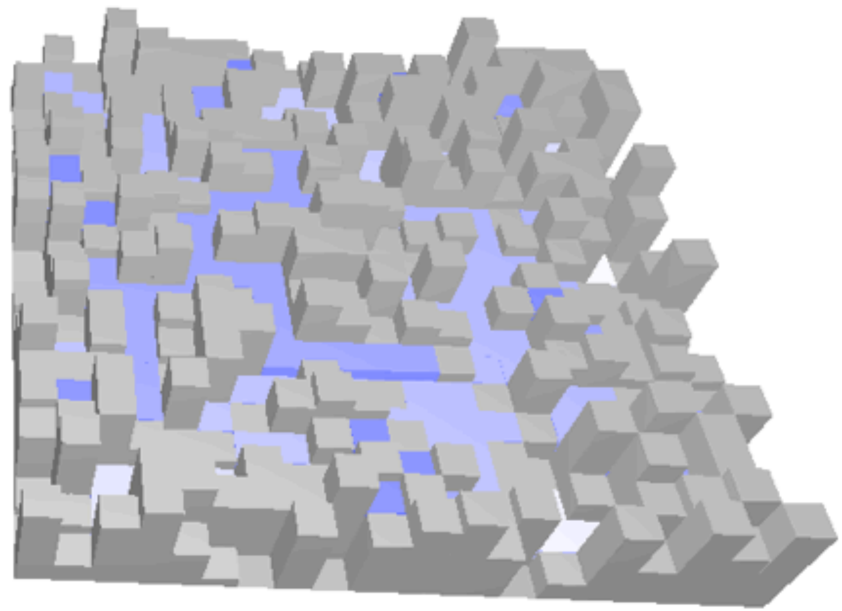
Retention on a 9-level system

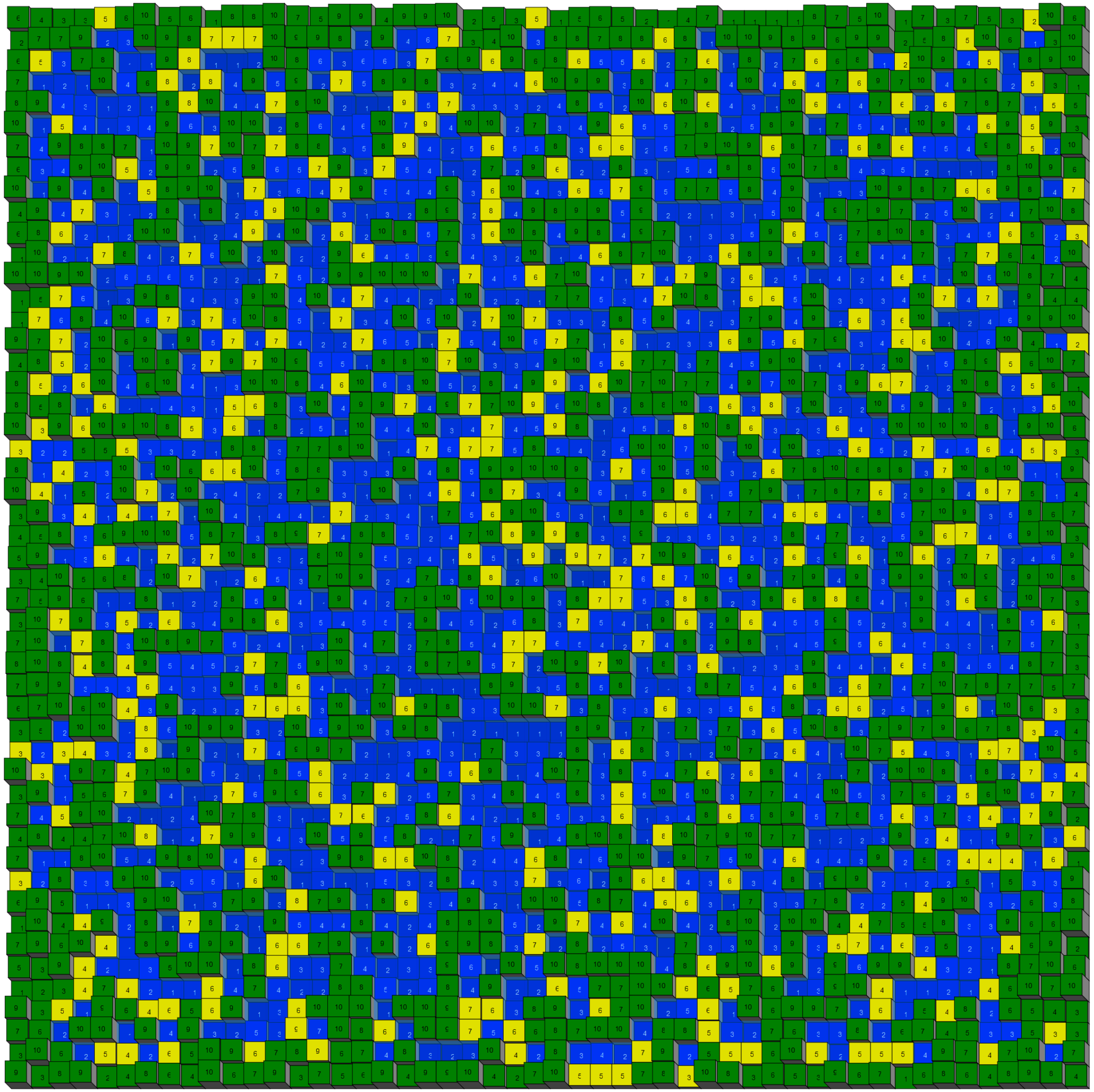
1,2, … 9



Water retained in a 10-level system



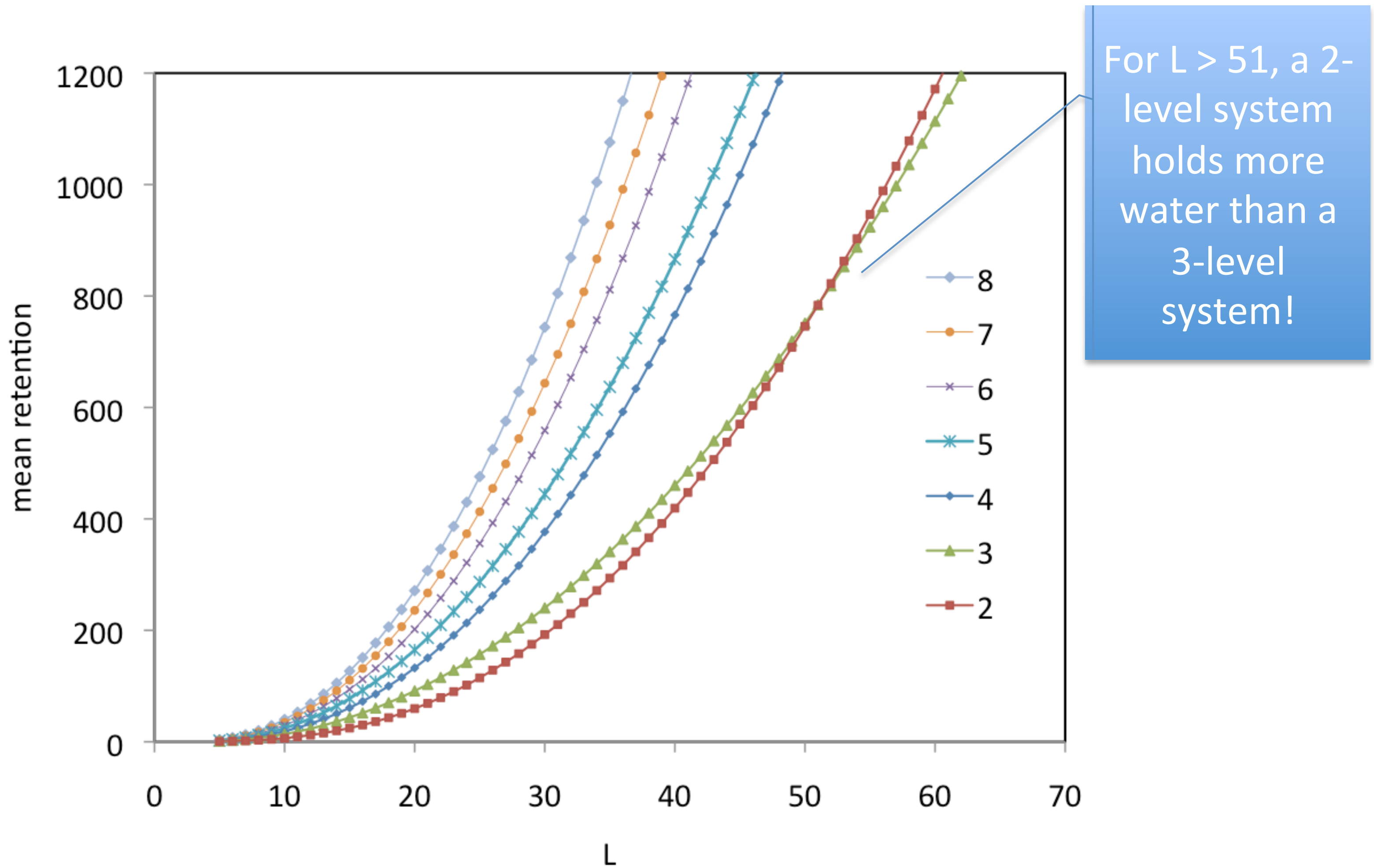




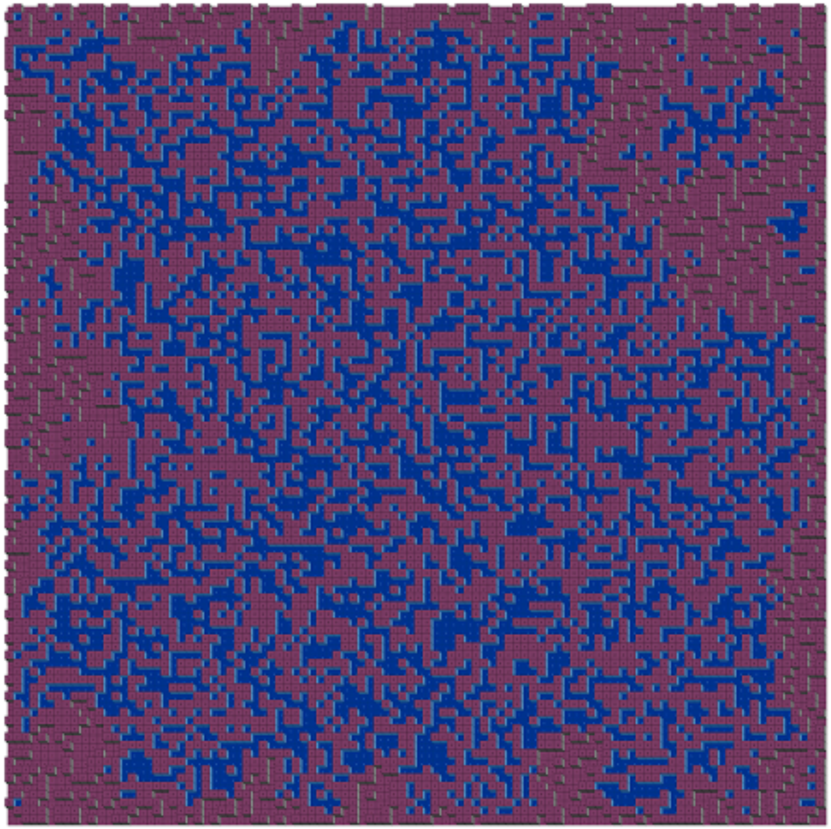
10-level system

- Blue sites: wet (retain water)
- Yellow sites are “spillway” sites which border on a pond and are the same level as the pond
- Green sites are dry (draining to a pond or to the boundary)

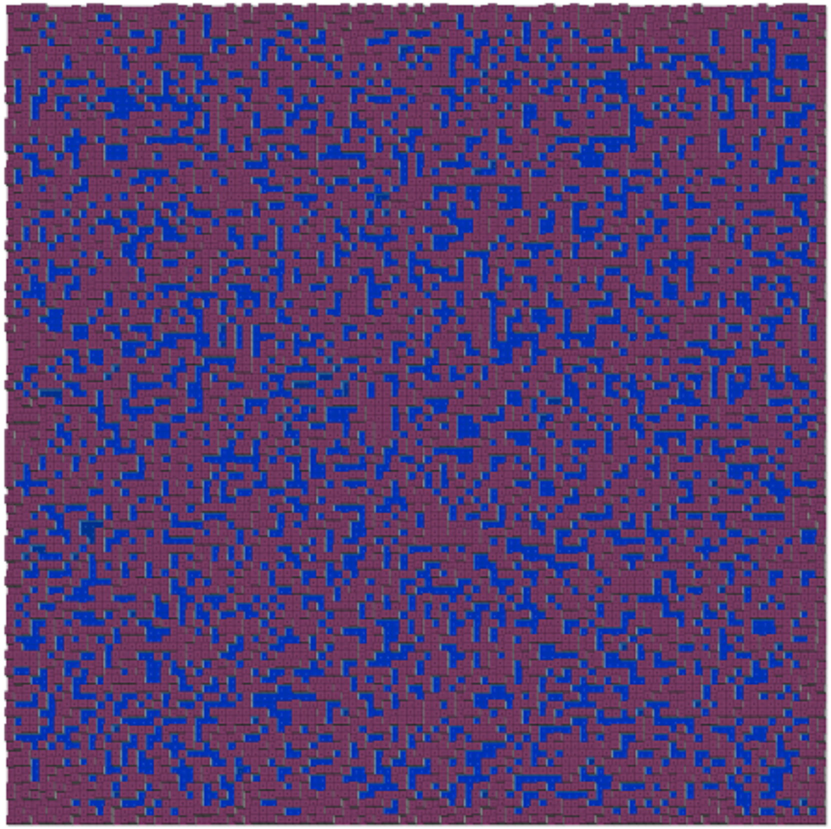
Retention as a function of system size L for n = 2, 3, 4, 5, 6, 7, 8 levels



2 vs. 3-level systems



Number range 1,2

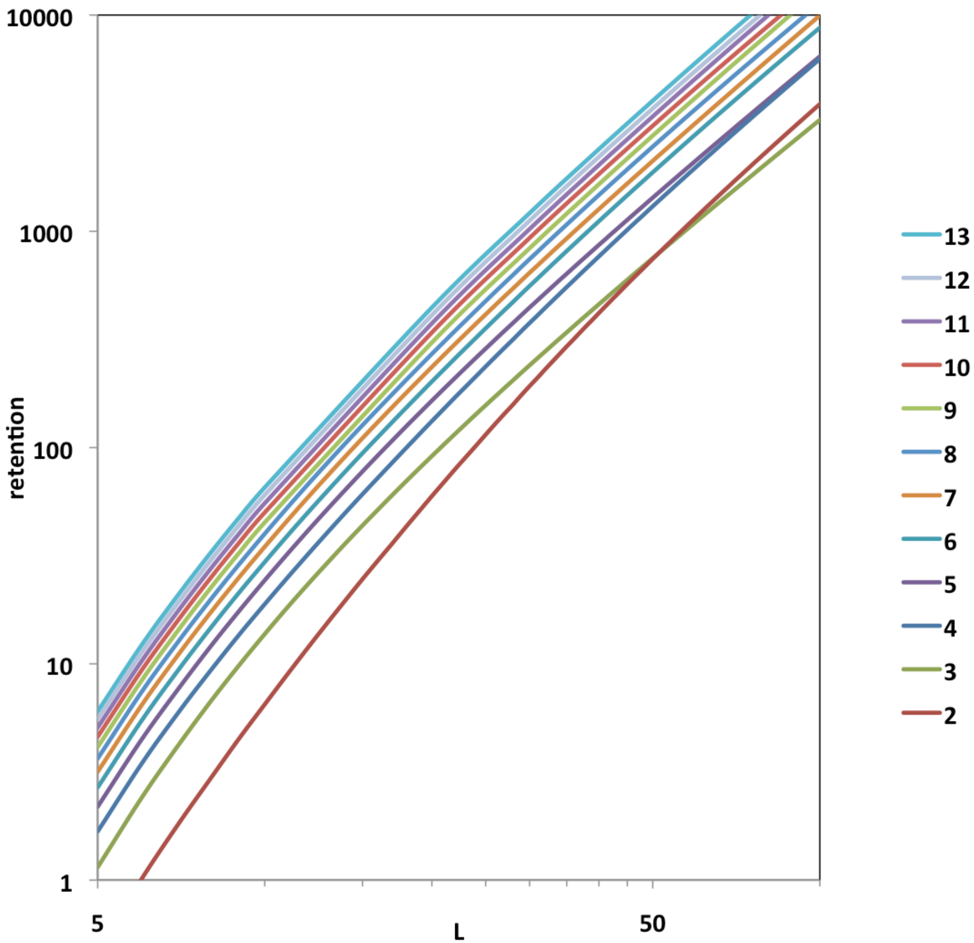


Number range 1,2,3

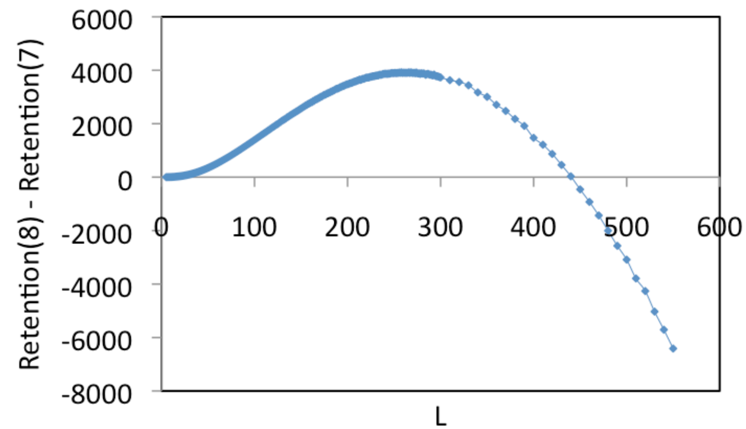
Note: by visual inspection the size of the lakes are larger in the 1,2 range
- the contribution made by cells retaining 2 units in the 1,2,3 range is negligible

More crossing curves for larger n

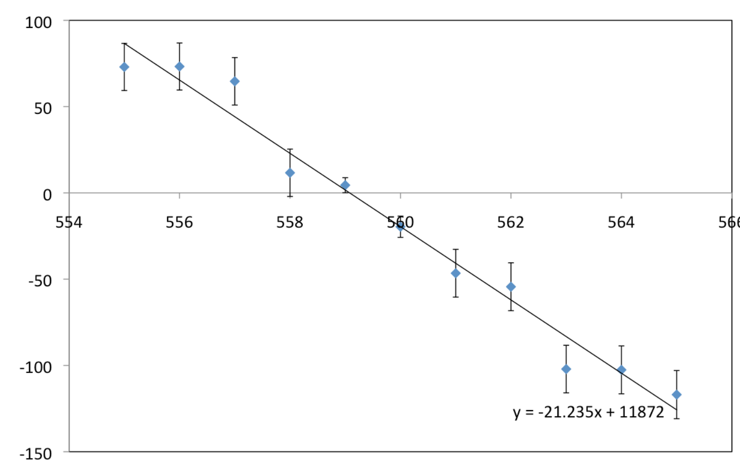
ln R vs. ln L:



R8 – R7



R9 – R10



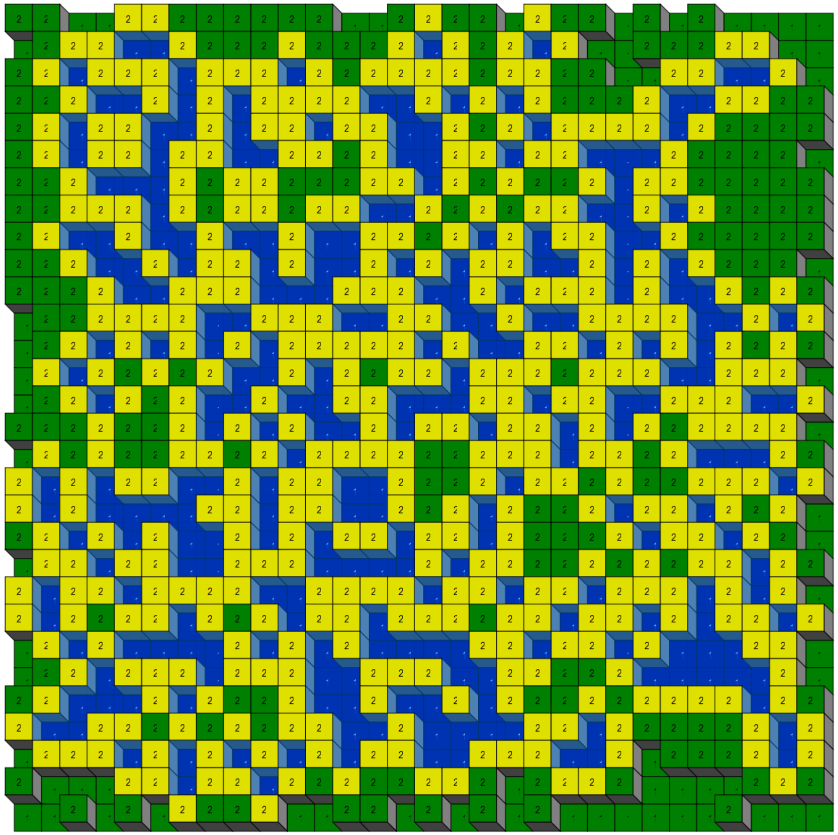
For small L, the curves are always ordered…

Crossing points:

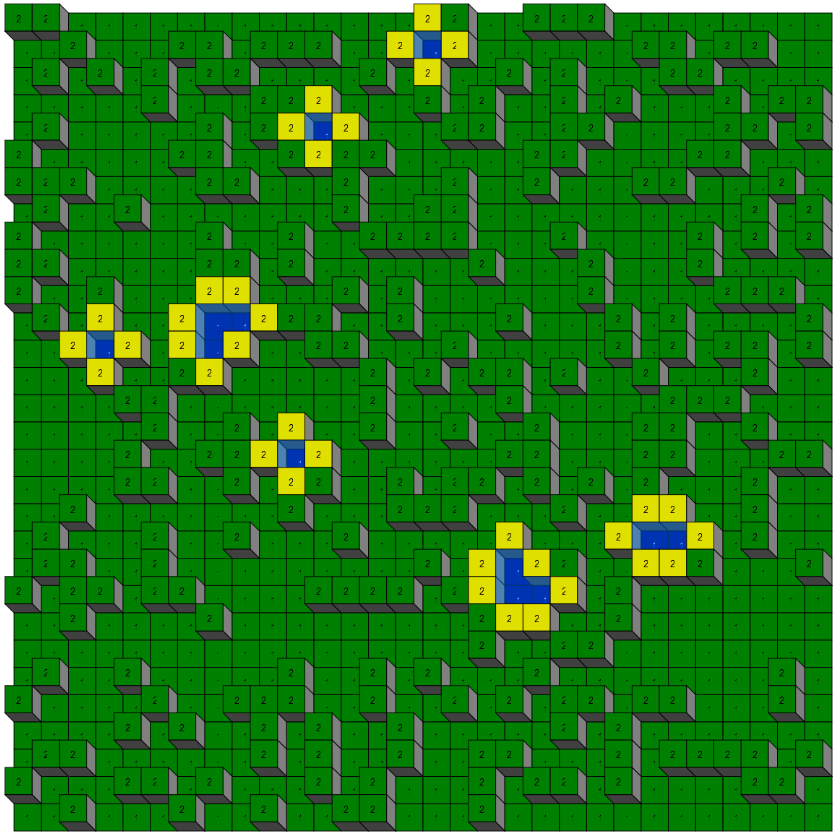
TABLE I. Crossing points where $R_n^{(L^*)} = R_{n+1}^{(L^*)}$, extrapolated to non-integer L^*.

n & $(n+1)$	L^*	$R_n^{(L^*)}$
2 & 3	51.12	790
4 & 5	198.1	26000
7 & 8	440.3	246300
9 & 10	559.1	502000
12 & 13	1390.6	4288500

Three-level system (equal probability)



Water on the lowest level
Same as retention on a 2-level
system with p = 1/3

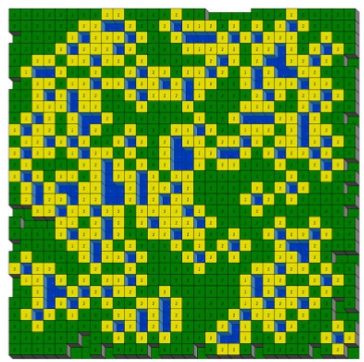


Water on the second level
Same as retention on a two-level
system with p = 2/3.

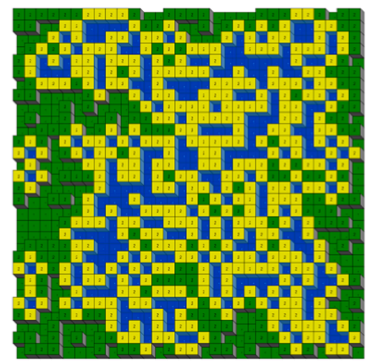
Thus, $R_3 = R_2(1/3)+R_2(2/3)$

The retention on a equal-probable 5-level system is the sum of the following four retentions on a two level system.

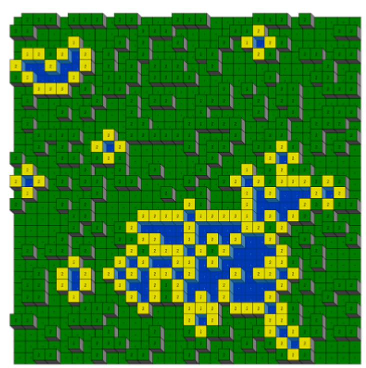
$R_5 = R_2(1/5)+R_2(2/5)+R_2(3/5)+R_2(4/5)$



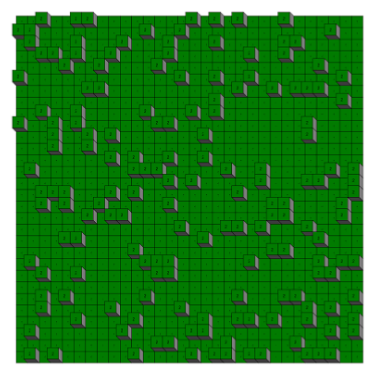
20 percent 1



40 percent 1

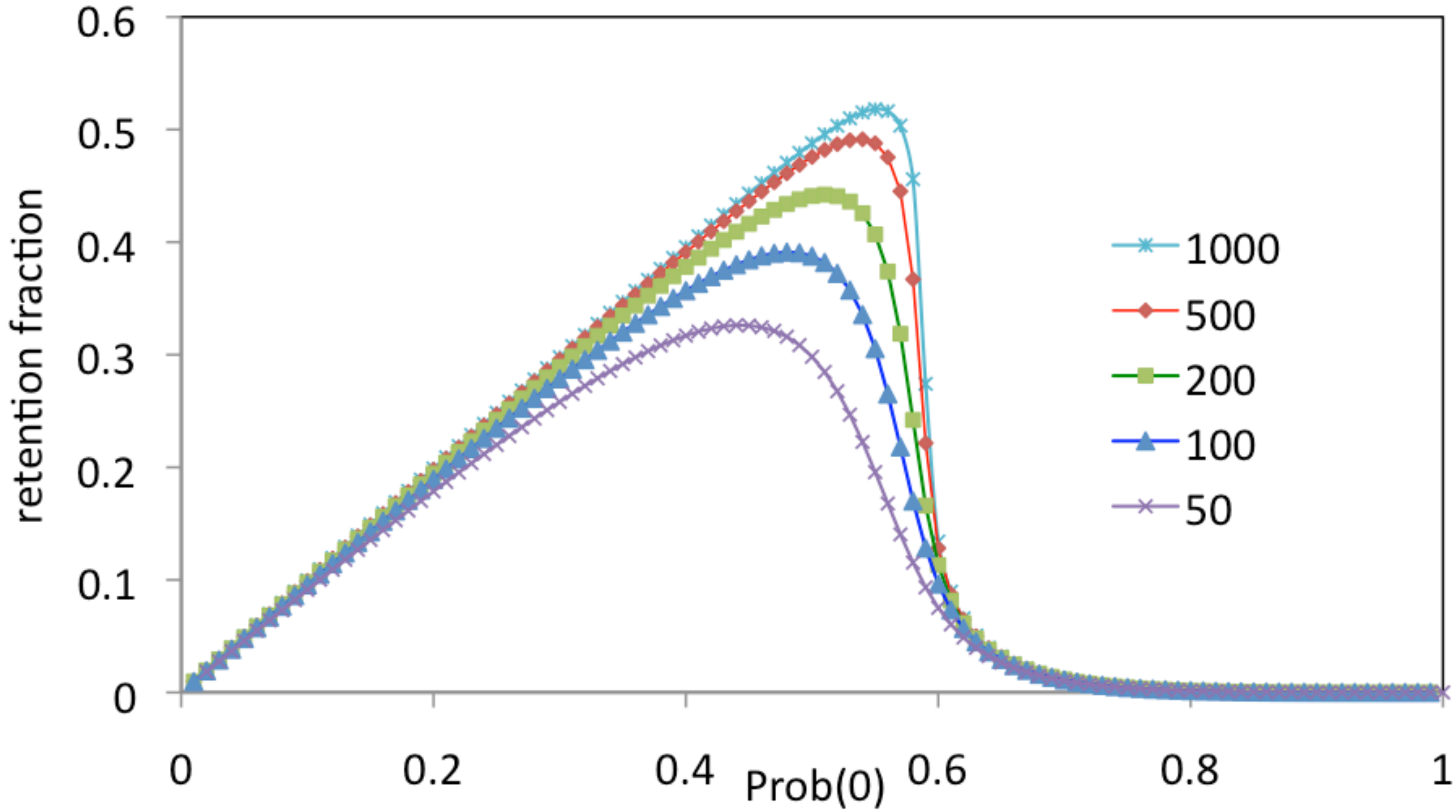


60 percent 1

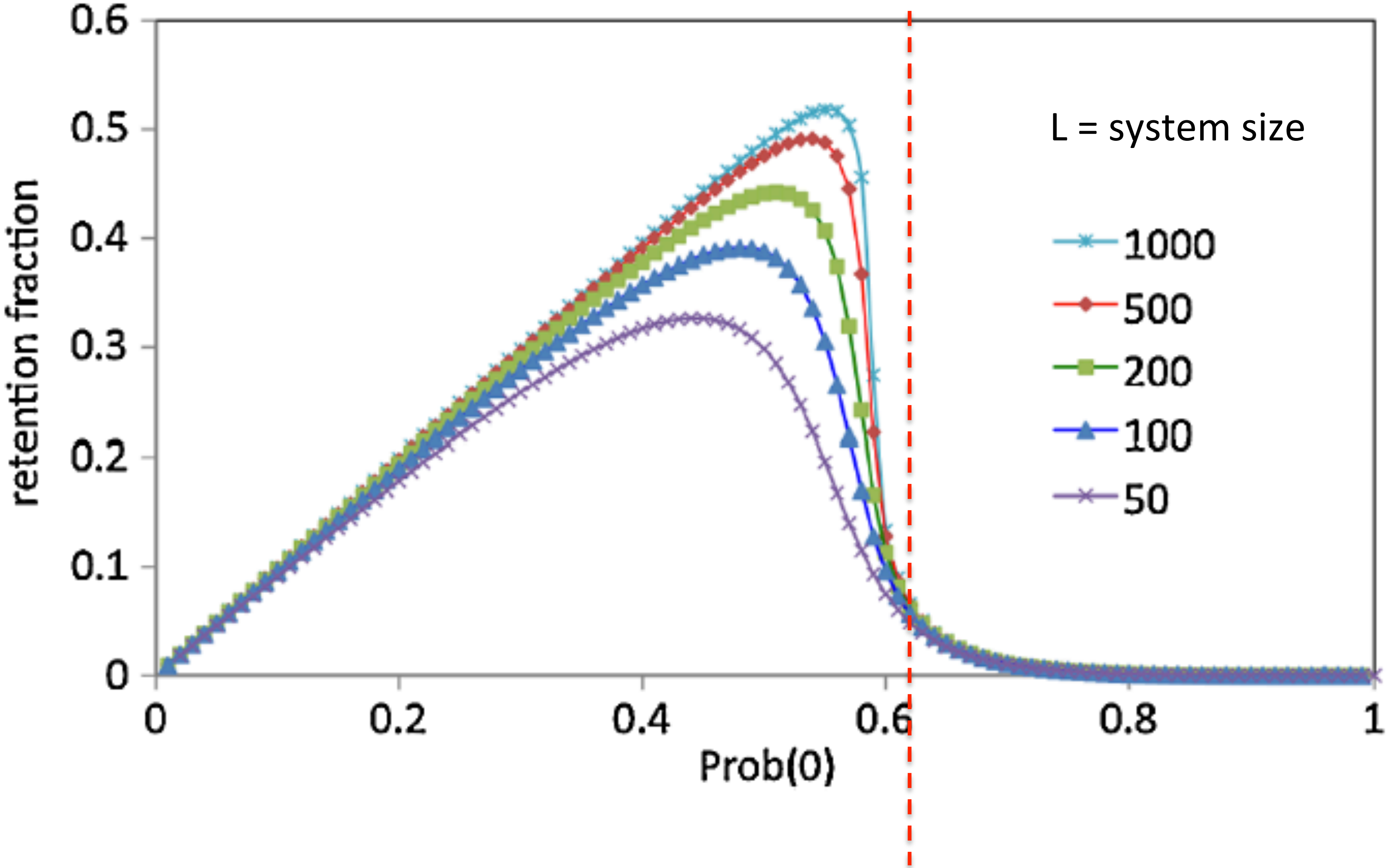


80 percent 1

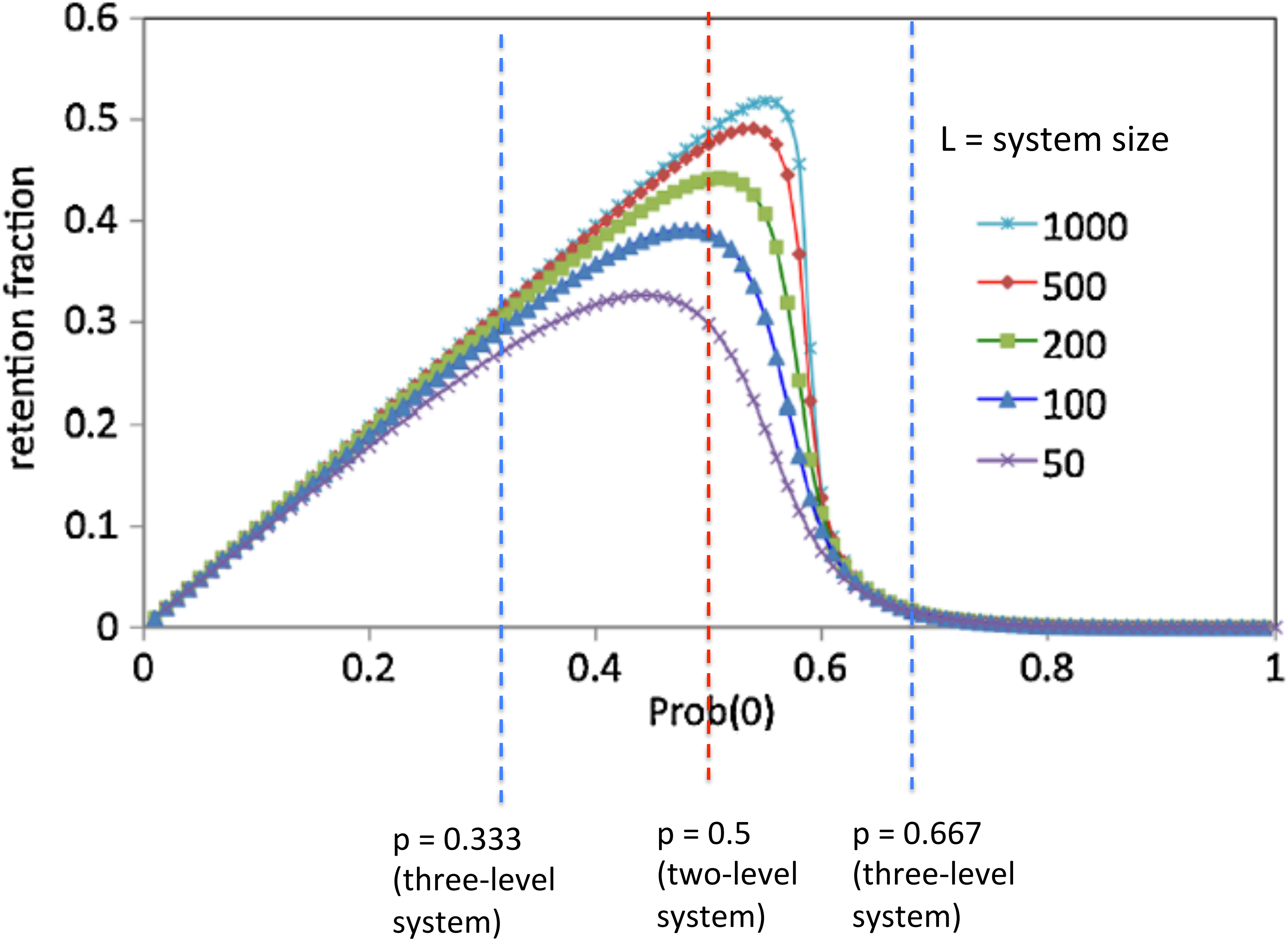
Retention of a 2-level system with with variable p = Prob(0)



Retention of a 2-level system with with variable p = Prob(0)



$p_c(\text{site}) = 0.592746$



0.6
0.5
0.4
0.3
0.2
0.1
0
retention fraction
0
0.2
0.4
0.6
0.8
1
Prob(0)
L = system size
1000
500
200
100
50
p = 0.333
(three-level
system)
p = 0.5
(two-level
system)
p = 0.667
(three-level
system)

Total retention of an n-level system can be related to the retention of 2-level systems of various occupations i/n for i = 1, 2, ... n-1:

$$R_n^{(L)} = \sum_{i=1}^{n-1} R_2^{(L)}\left(\frac{i}{n}\right)$$

Or, in the continuum limit where n goes to infinity, the relative retention per site follow from

$$r = \int_0^1 r_2(p)dp$$

where r = $R_n/(n\ L^2)$ and r = $R_n/(n\ L^2)$

Likewise, one can find the total number of wet sites from R_2: (1/i)R(i/n) is equal the number of sites with retention i. Thus, the total number of wet sites is

$$W_n^{(L)} = \sum_{i=1}^{n-1} \frac{1}{i} R_2^{(L)}\left(\tfrac{i}{n}\right)$$

Or, in the continuum limit

$$w = \int_0^1 (r_2(p)/p)dp$$

where $w = W/L^2$ and $r_2 = R_2/L^2$. This give $w = p_c$ + a little bit.

$$r_2(p) = p - P_\infty(p)$$

- Then we can relate r to an integral of $P_\infty(p)$

$$r = \int_0^1 (p - P_\infty(p))dp = 1/2 - \int_0^1 P_\infty(p)dp$$

- And likewise for w, providing an interpretation of integrals of $P_\infty(p)$ for the first time!

Proof that retentions are a monotonic function of n for small L

- 3x3 system: can solve exactly:

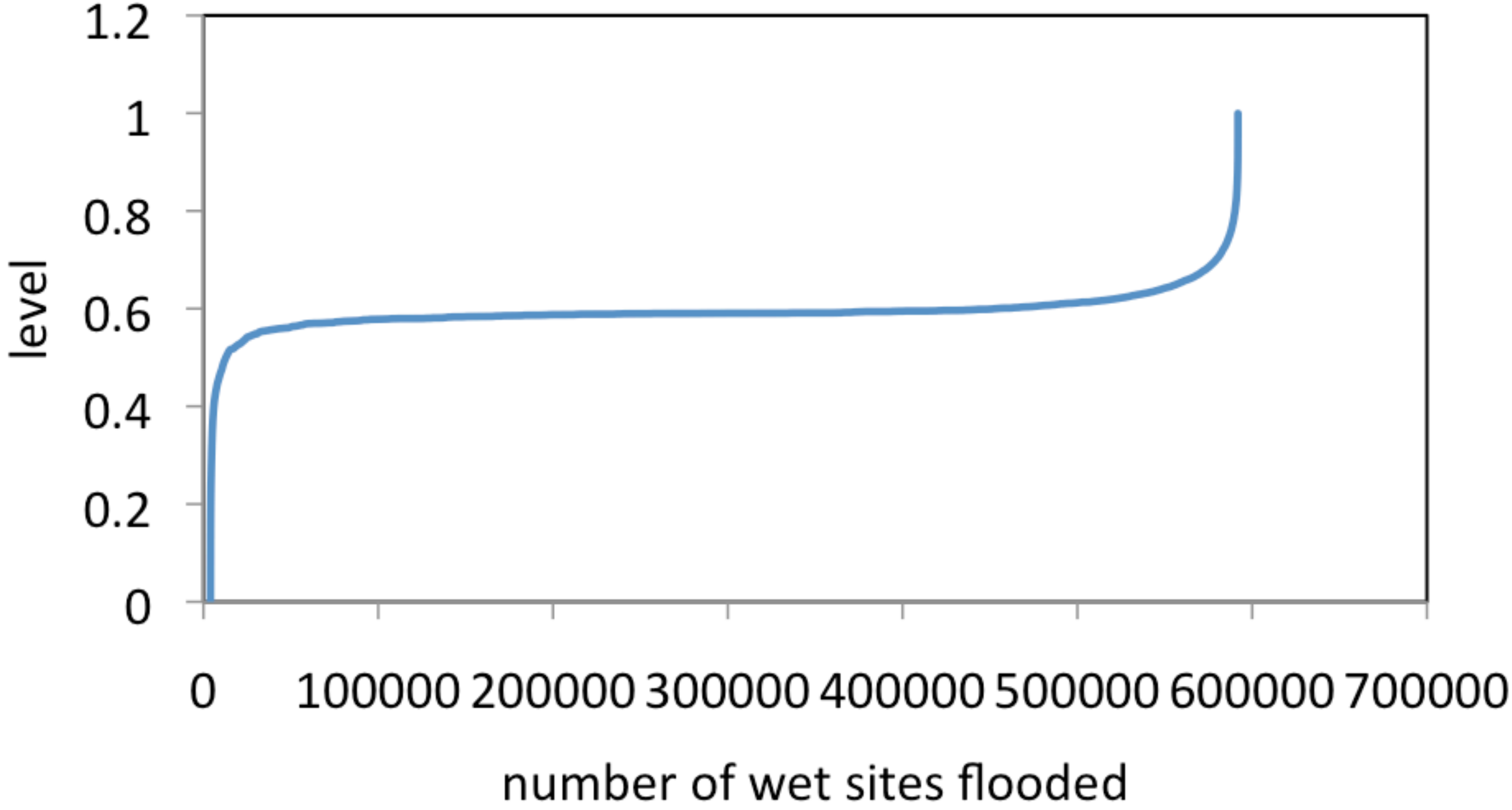
$$R_n^{(3)} = (n^2 - 1)(3n^2 - 2)/(60n^5)$$

- Which is monotonic in n. Thus, for n such that $R_n > R_{n+1}$, there must be crossing at some value of L.

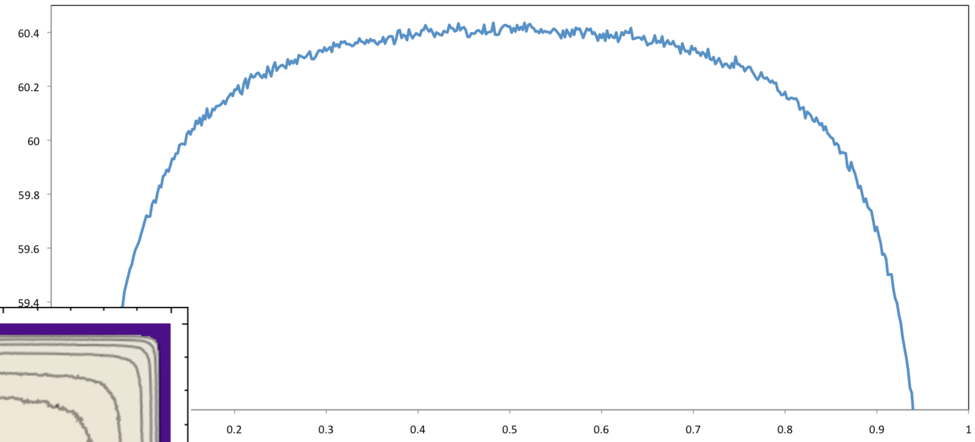
Algorithm to find water levels

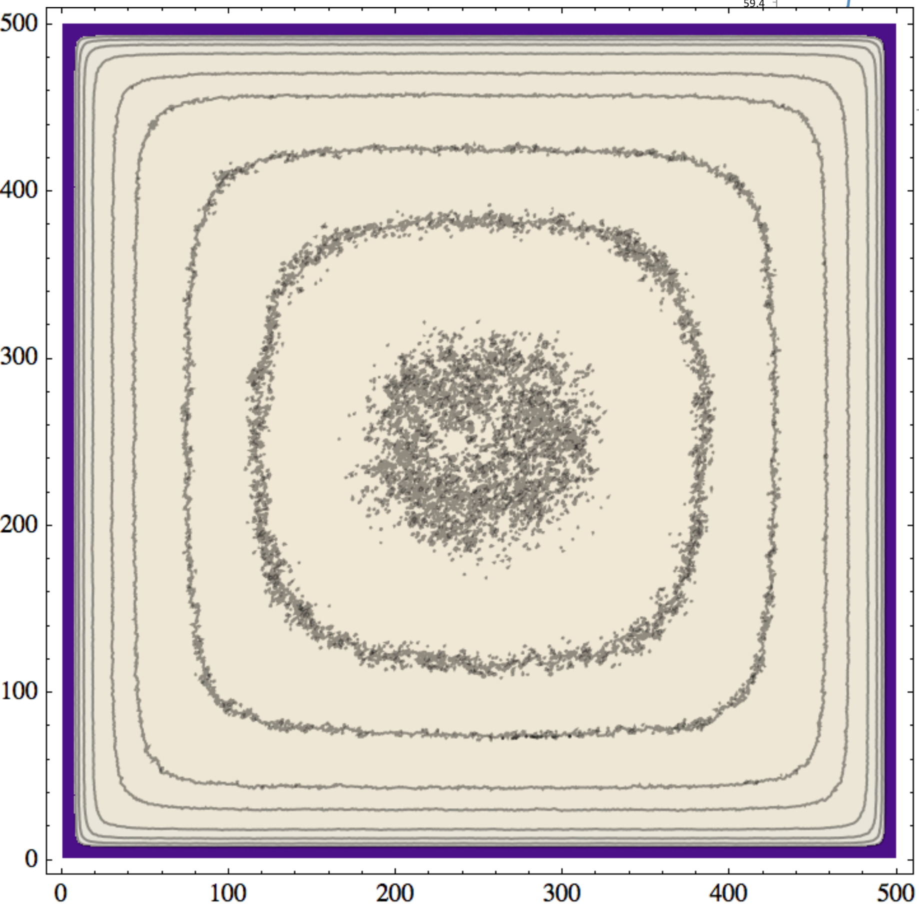
- Invade fluid from the outside boundary, always choosing the site with the lowest terrain height, while monotonically increasing water level at the boundaries.
- Remember the water level at each site when it was first flooded
- Retention equals flooding level minus terrain height.

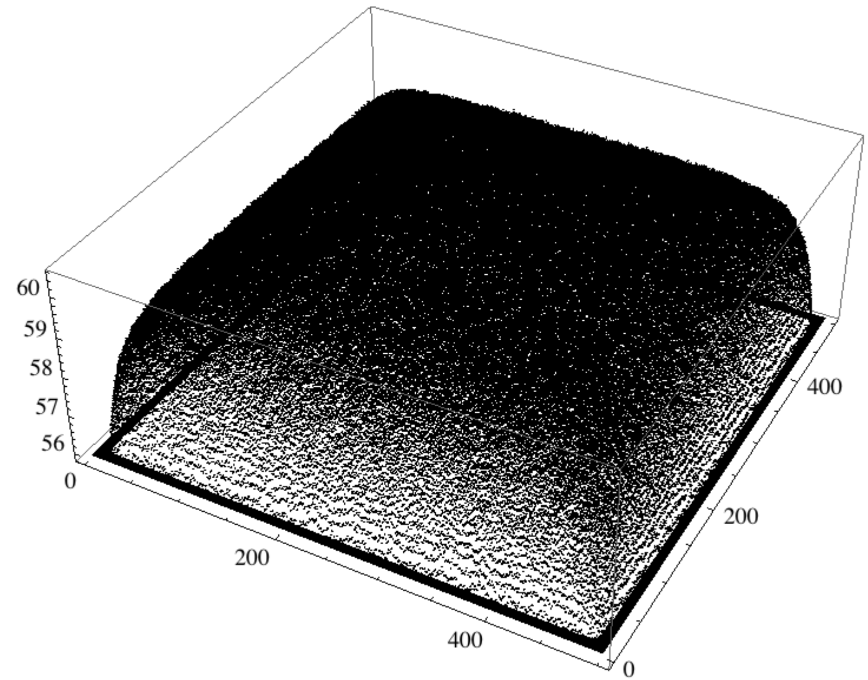
Level vs. the number of sites flooded during the algorithm



Average level of wet sites only:
Reaches about 0.604 near the center…







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Retention Capacity of Random Surfaces

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