Advanced Percolation Algorithms

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DPG - School on Physics

Efficient Algorithms in Computational Physics

10 - 14 September 2012, Physikzentrum Bad Honnef, Germany

Alexander K. Hartmann (University of Oldenburg) and
A. Peter Young (University of California Santa Cruz)

http://www.dpg-physik.de/dpg/pbh/aktuelles/S112.html?lang=en&

DPG

Percolation

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John Hammersley
21 March 1920 - 2 May 2004

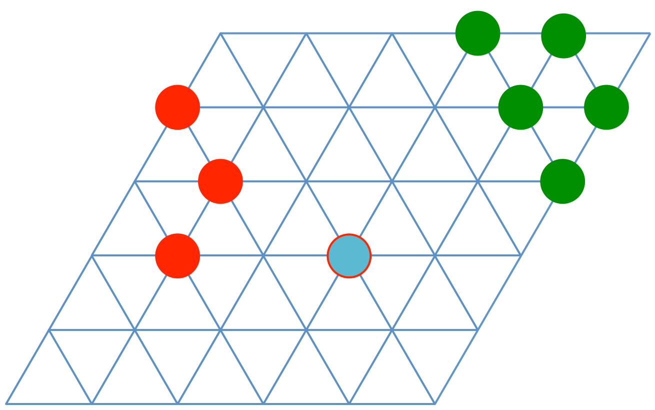
Dietrich Stauffer (Cologne)
In Niterói, Brazil on 60th birthday.

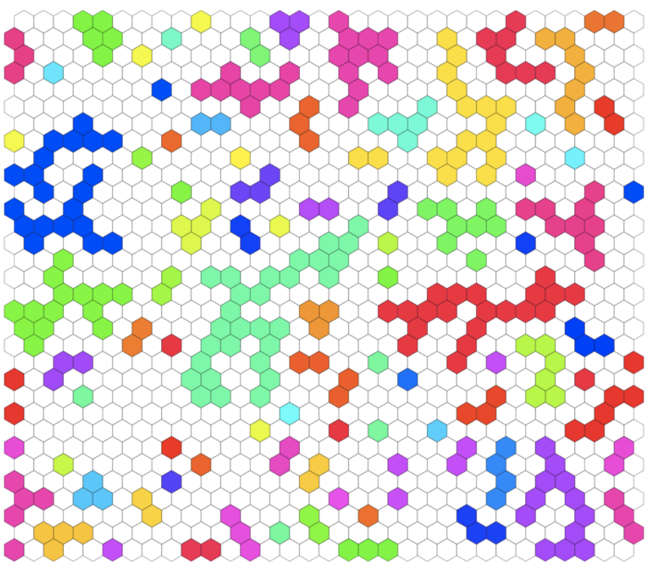
Percolation

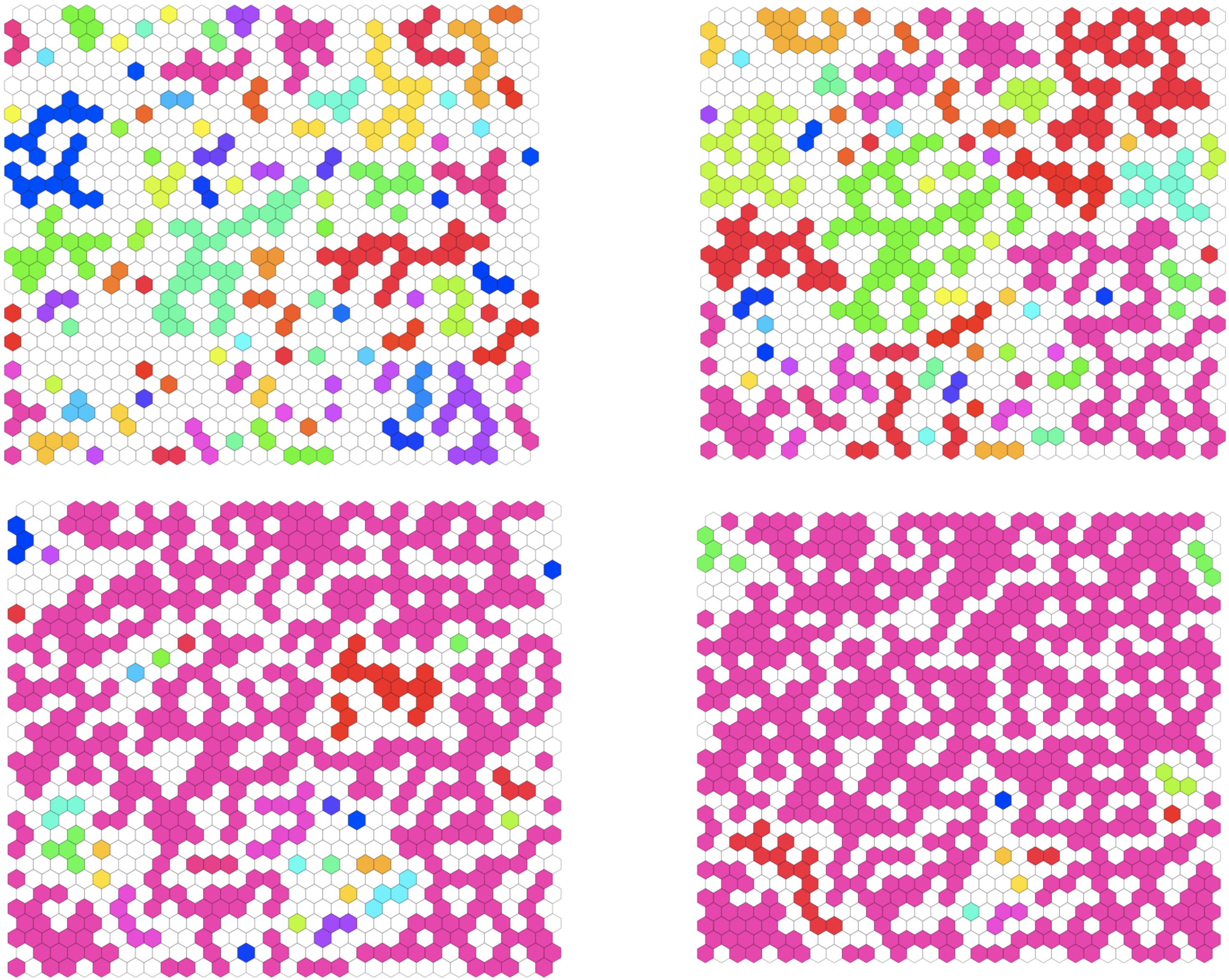
- Emergence of connectivity in random systems
- One of the simplest statistical mechanical models showing critical behavior
- Applications to studies of porous media, random networks, epidemiology, polymerization, electrical conductivity

Site percolation

- Site percolation on the triangular lattice = tiling of hexagons on a honeycomb lattice
- Let *p* = probability that a site or tile is "occupied"



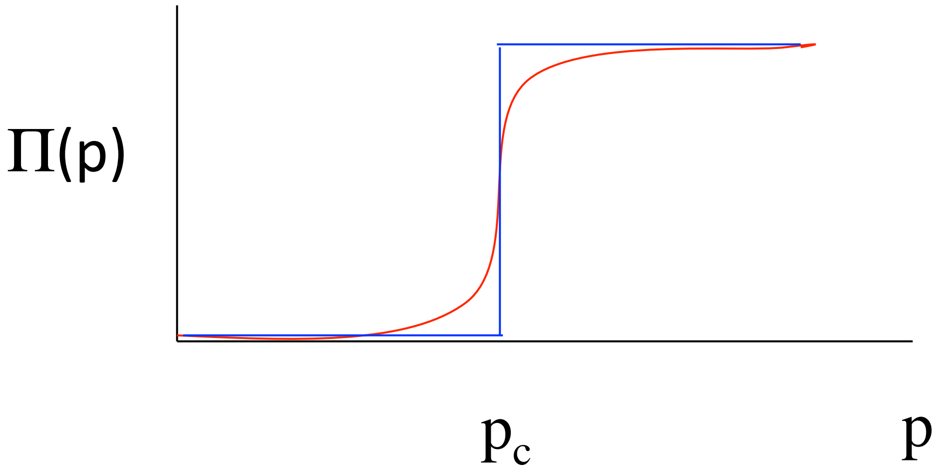


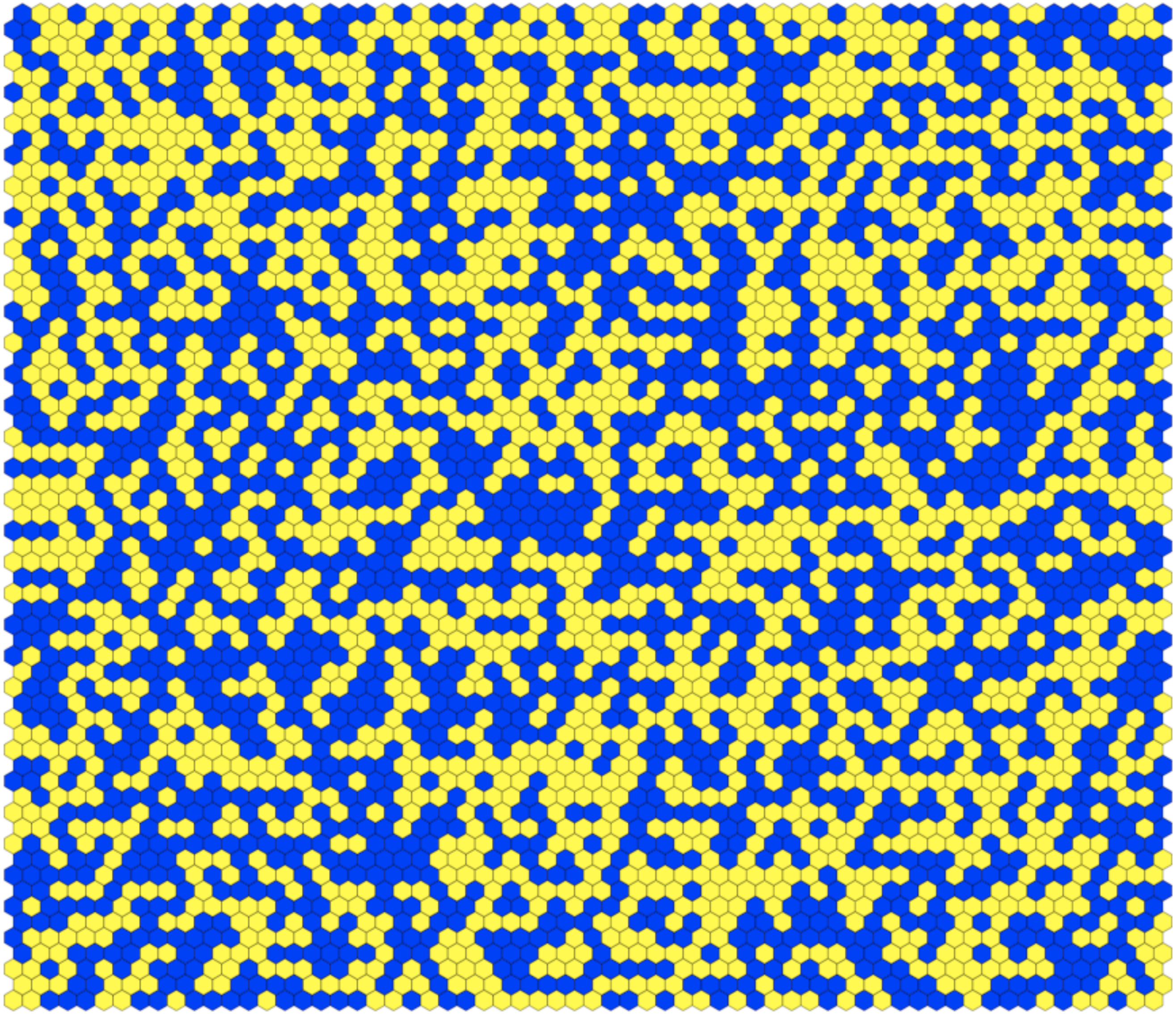


- p = 0.3, 0.4, 0.5, 0.6, showing clusters of occupied tiles. At p = 0.5, a cluster "percolates" (crosses, or wraps around for periodic b.c.).

Percolation probability

- Let $\Pi(p)$ be the probability that the system percolates. (In this case, that can mean wrapping around).
- As the system size L goes to infinity, $\Pi(p)$ becomes a step function and defines the critical threshold p_c.

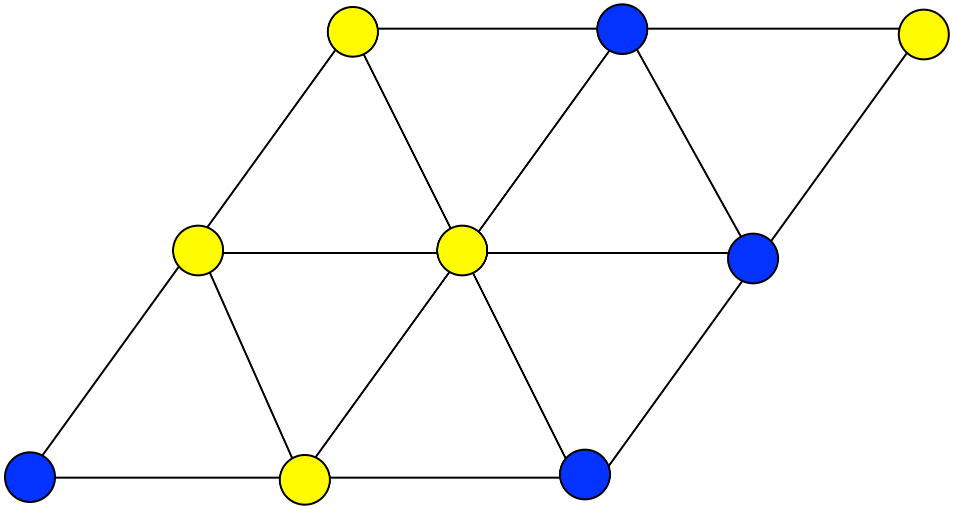




- System at threshold, $p = 0.5$

Percolation threshold

- By symmetry, p_c equals 1/2 for the hexagonal tiling.
- This system is equivalent to *site percolation* on a triangular lattice:



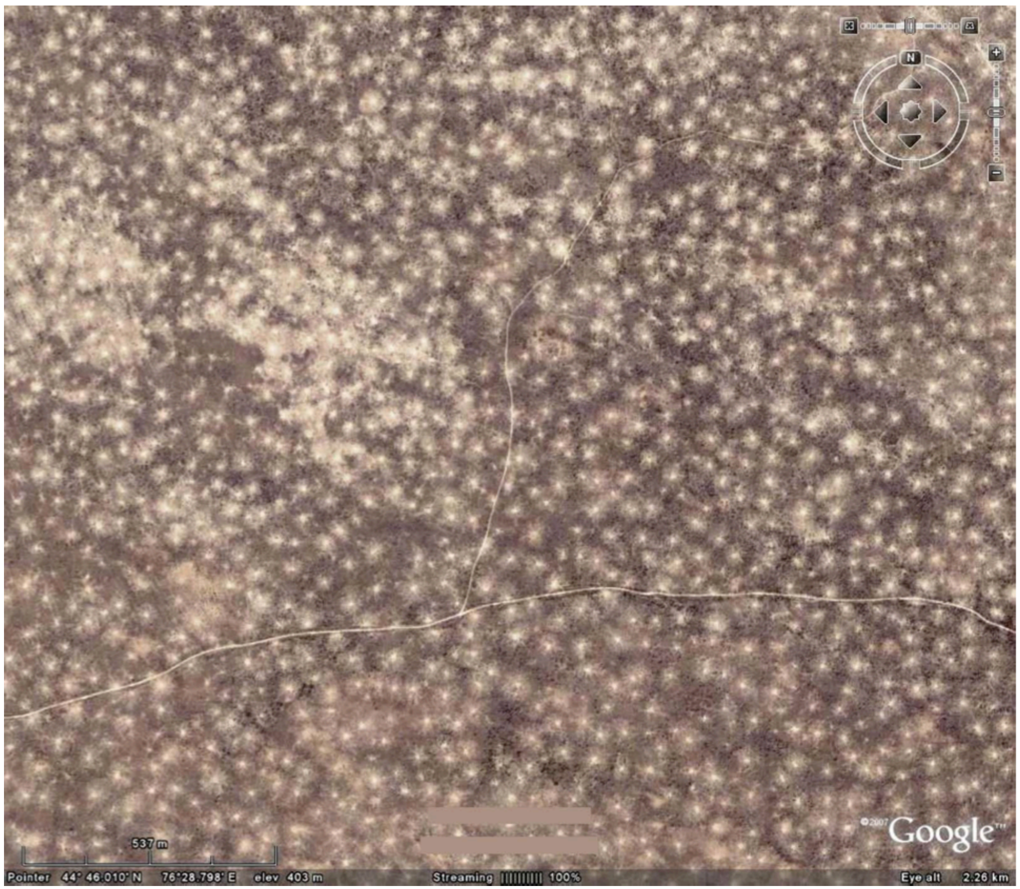
nature Vol 454|31 July 2008|doi:10.1038/n

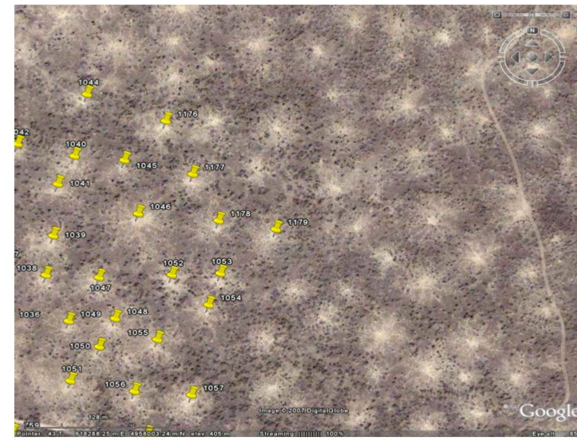
LETTERS

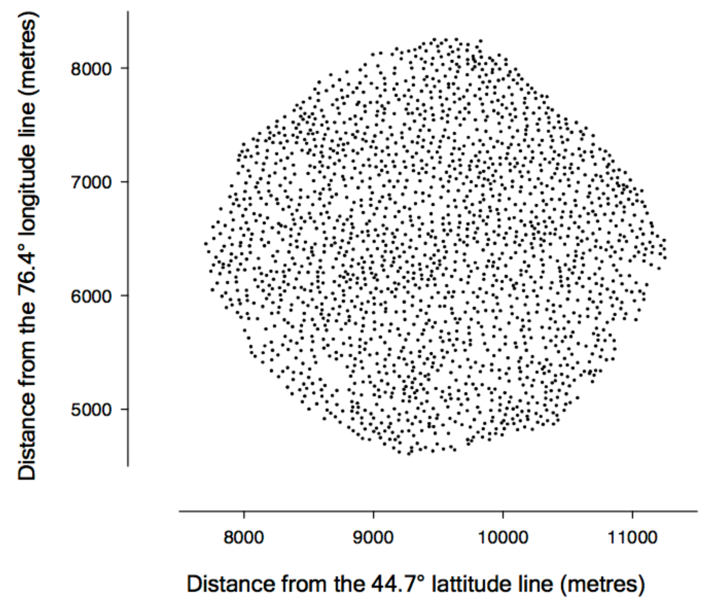
The abundance threshold for plague as a critical percolation phenomenon

S. Davis[1], P. Trapman[2], H. Leirs[3,4], M. Begon[5] & J. A. P. Heesterbeek[1]

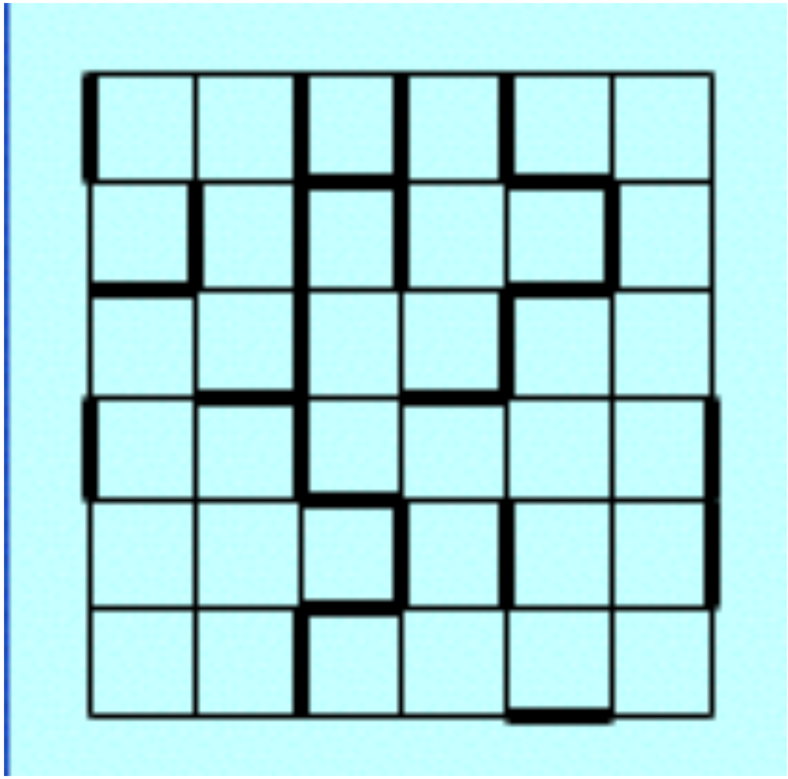
Prairie dog colonies viewed from space (Google map)







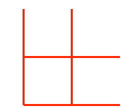
Bond percolation on the square lattice:

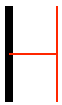


Bonds are made "occupied" (conducting) with an independent probability p

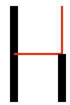
Sites connected by bonds form *clusters*.

Exact enumeration on 2 x 3 system (16 percolating diagrams out of $2^5 = 32$ total)

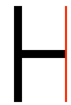




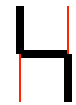
$2p^2q^3$



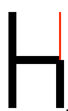
$4p^3q^2$



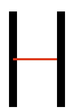
$2p^3q^2$



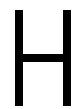
$2p^3q^2$



$4p^4q$



p^4q

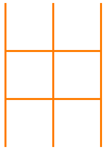


p^5

$$\begin{aligned}\Pi_2(p) &= 2p^2q^3 + 8p^3q^2 + 5p^4q + p^5 \\ &= 2p^2 + 2p^3 - 5p^4 + 2p^5\end{aligned}$$

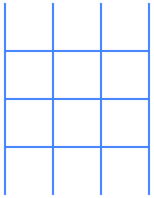
L x (L+1) system, bond percolation

L = 3   13 bonds 2^{13} graphs



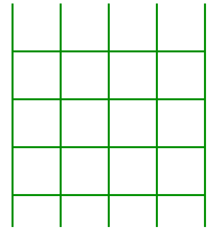
$$\Pi_3(p) = p^{13} + 13p^{12}q + 78p^{11}q^2 + 283p^{10}q^3 + 677p^9q^4 + 1078p^8q^5 + 1089p^7q^6 + 627p^6q^7 + 209p^5q^8 + 38p^4q^9 + 3p^3q^{10}$$

L = 4   25 bonds 2^{25} graphs



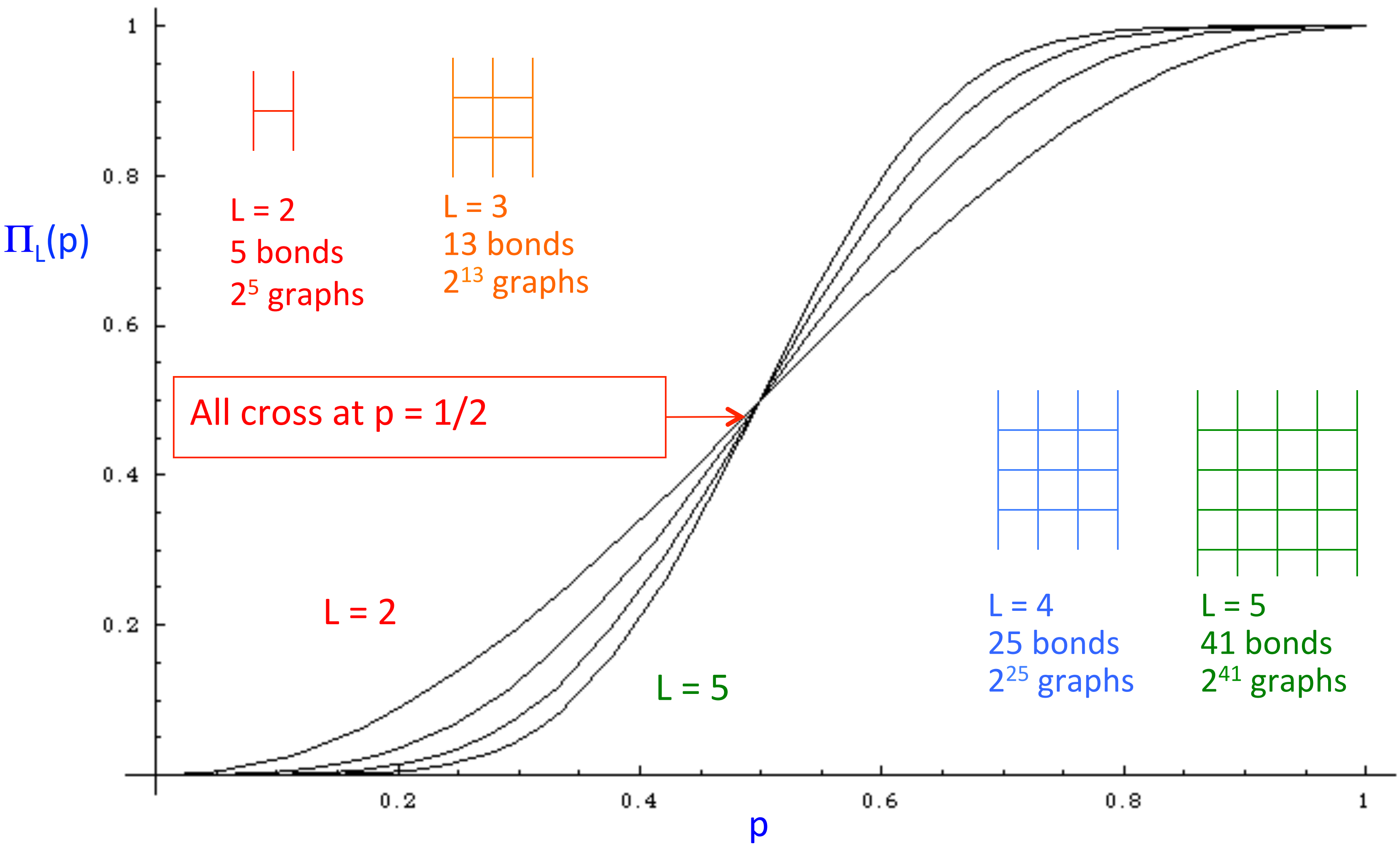
$$\Pi_4(p) = p^{25} + 25qp^{24} + 300q^2p^{23} + 2300q^3p^{22} + 12646q^4p^{21} + 53028q^5p^{20} + 175870q^6p^{19} + 471428q^7p^{18} + 1032857q^8p^{17} + 1854463q^9p^{16} + 2715264q^{10}p^{15} + 3204984q^{11}p^{14} + 3001802q^{12}p^{13} + 2198498q^{13}p^{12} + 1252416q^{14}p^{11} + 553496q^{15}p^{10} + 188512q^{16}p^9 + 48718q^{17}p^8 + 9272q^{18}p^7 + 1230q^{19}p^6 + 102q^{20}p^5 + 4q^{21}p^4$$

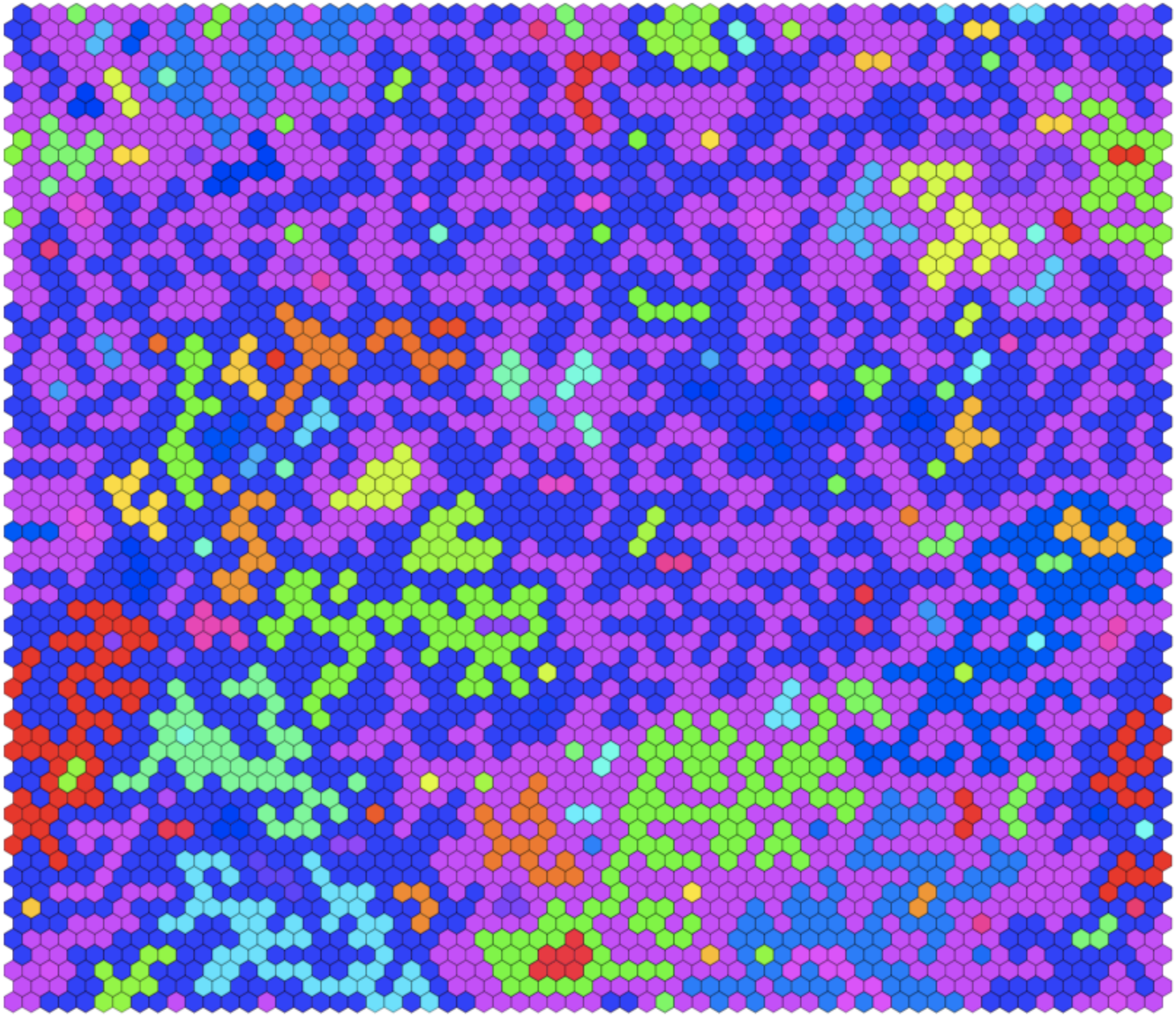
L = 5 41 bonds 2^{41} graphs



$$\Pi_5(p) = p^{41} + 41qp^{40} + 820q^2p^{39} + 10660q^3p^{38} + 101270q^4p^{37} + 749393q^5p^{36} + 4496176q^6p^{35} + 22477578q^7p^{34} + 95490323q^8p^{33} + 349786536q^9p^{32} + 1116971463q^{10}p^{31} + 3134969579q^{11}p^{30} + 7779300450q^{12}p^{29} + 17133827010q^{13}p^{28} + 33563042210q^{14}p^{27} + 58491129097q^{15}p^{26} + 90564837046q^{16}p^{25} + 124238149114q^{17}p^{24} + 150402767321q^{18}p^{23} + 159936494743q^{19}p^{22} + 148703966218q^{20}p^{21} + 120424971002q^{21}p^{20} + 84726175457q^{22}p^{19} + 51709873279q^{23}p^{18} + 27346331336q^{24}p^{17} + 12512609660q^{25}p^{16} + 4941145799q^{26}p^{15} + 1677110510q^{27}p^{14} + 486249350q^{28}p^{13} + 119354470q^{29}p^{12} + 24492389q^{30}p^{11} + 4127945q^{31}p^{10} + 557029q^{32}p^{9} + 57922q^{33}p^{8} + 4362q^{34}p^{7} + 212q^{35}p^{6} + 5q^{36}p^{5}$$

Π_L vs. p for bond percolation on a square lattice





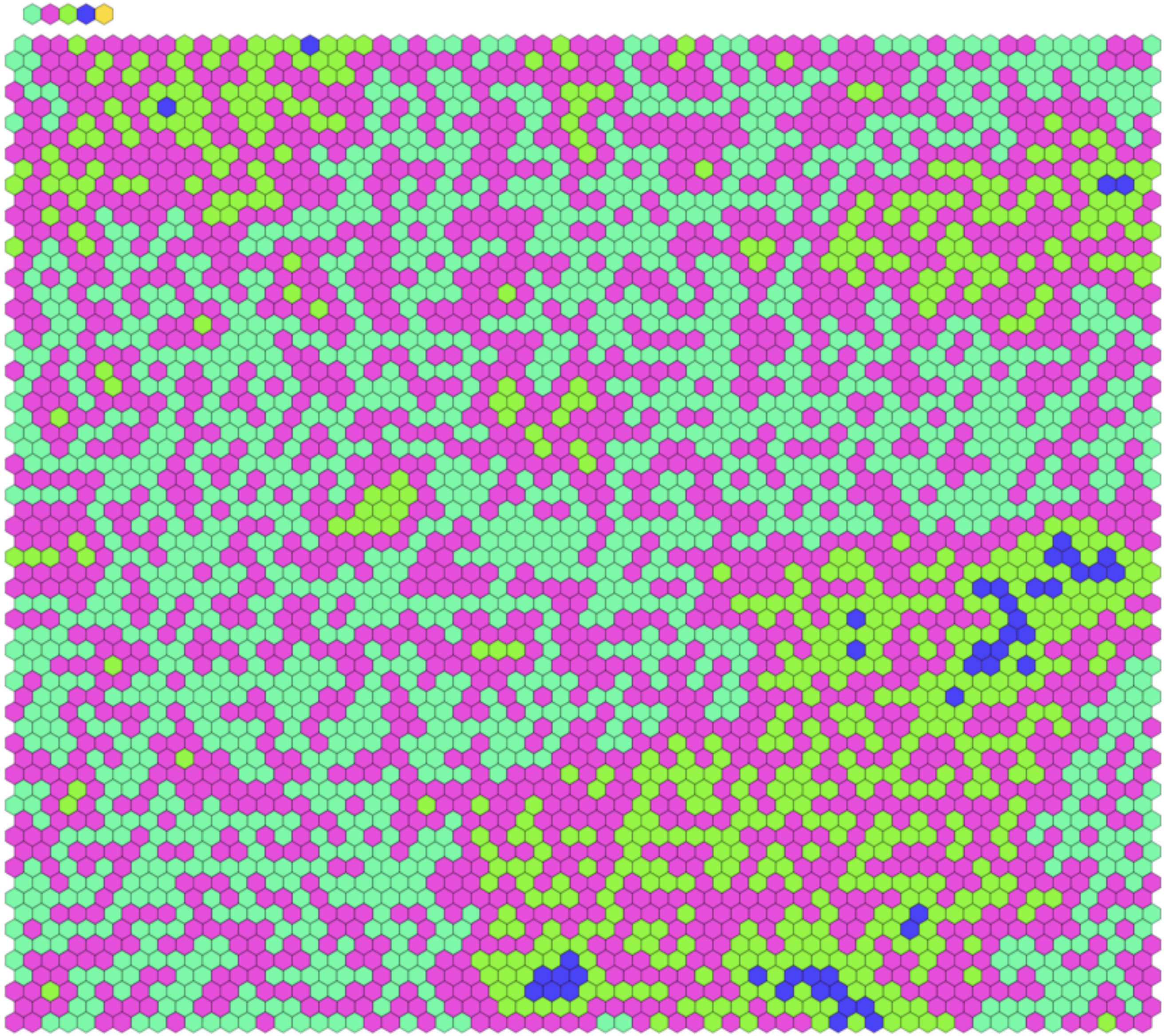
- *p* = 0.5, clusters of both yellow and blue tiles are shown (periodic b.c.)

Conventional form of the size distribution:

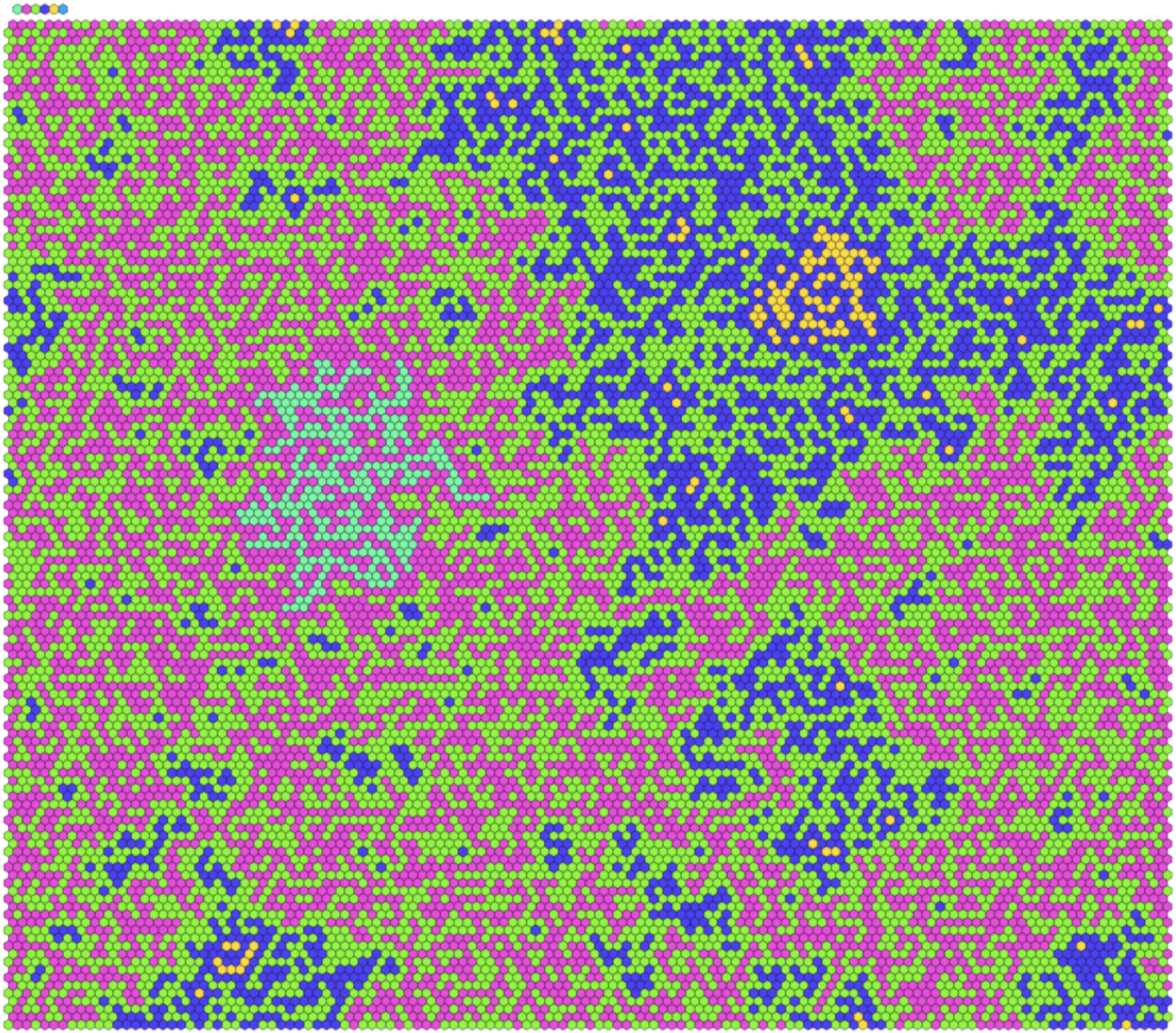
- Number of clusters (per site of the lattice) of size *s* at p_c:

$$n_s(p_c) \sim A s^{-\tau}$$

- Here τ = 187/91 for all 2-d systems, but *A* varies from system to system and is thus non-universal.



- Depth from a single point is shown in different colors - maximum depth is 4



- 128x128 - max depth = 5

For any system, at the critical threshold, we have

$$\text{Average depth} = 4C \ln \frac{L}{a}$$

Where *L* = system size, *a* = tile size, and

$$C = \frac{1}{8\sqrt{3}\pi} = 0.022972037...$$

J. L. Cardy and R. Z.., 2003

The Enclosed-Area Distribution

- This implies that the number of clusters (or hulls) whose enclosed area is greater or equal to a value *A* is given by

$$\mathrm{Nr}(\geq A) \sim \frac{CL^2}{A}$$

valid for any system at criticality.

Zipf's law form for 2d critical clusters (p_c)

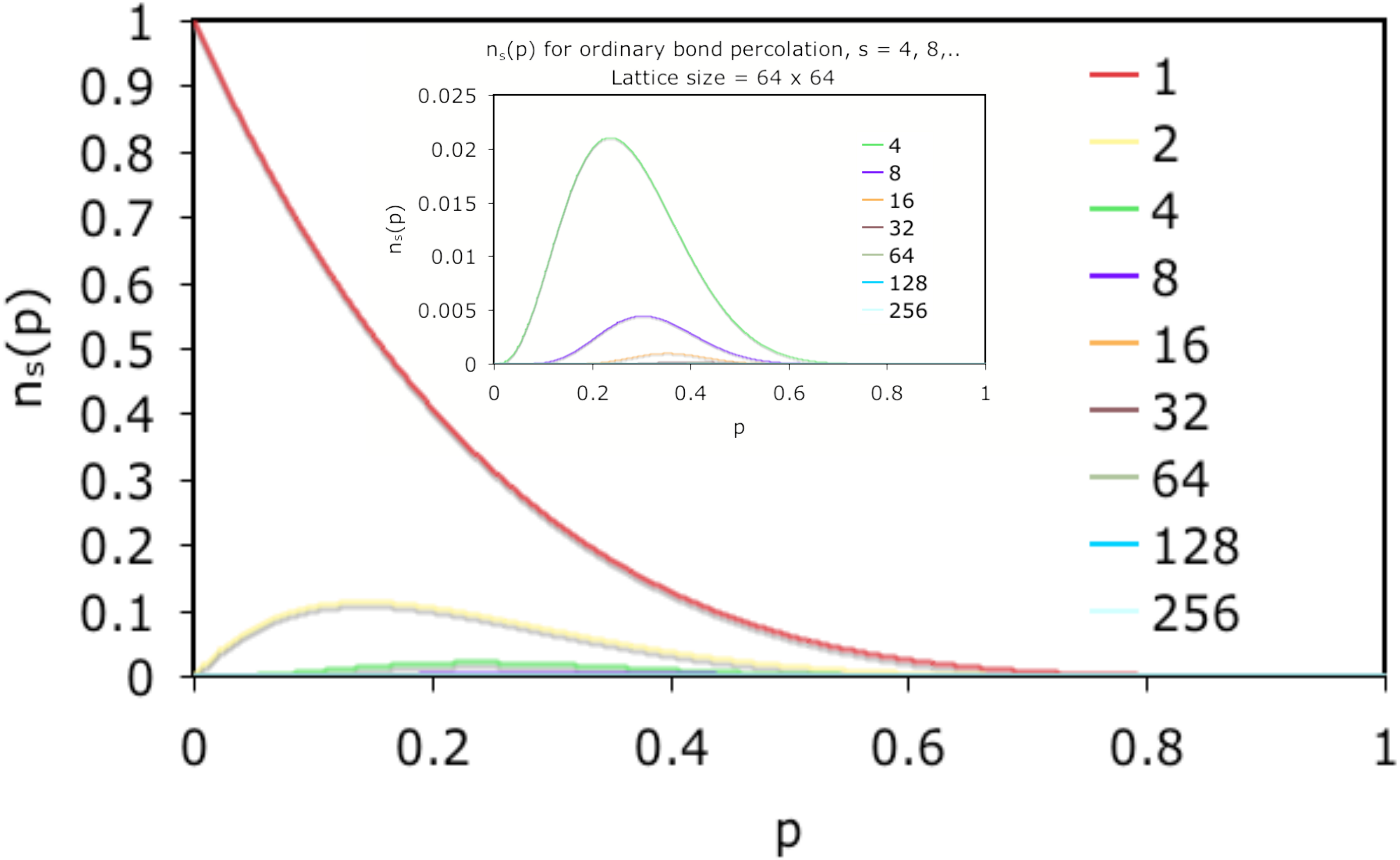
- If you rank-order all cluster areas, largest to smallest, then for large *n*, the enclosed hull area A_n of the *n*-th largest cluster is given by the fully universal formula:

$$\frac{A_n}{L^2} \sim \frac{C}{n}$$

- Where $C = 1/(8\pi\sqrt{3}) = 0.0229...$ and L = system dimension. (Ziff, Lorenz & Kleban 1999, Cardy & Ziff, 2003)

J. L. Cardy and R. Z.., 2003

$n_s(p)$ for ordinary bond percolation, s = 1, 2, 4, ..
Lattice size = 64 x 64



- Scaling in percolation:

$$n_s(p) \sim As^{-\tau} f(B(p - p_c)s^{1/\sigma})$$
$$\text{as } s \to \infty, p \to p_c, (p - p_c)s^{1/\sigma} = \text{const.}$$

Where τ, σ, and f(z) are *universal*, while A and B are *non-universal* metric factors

(In 2d, $\tau = 187/91$, $\sigma = 36/91$)

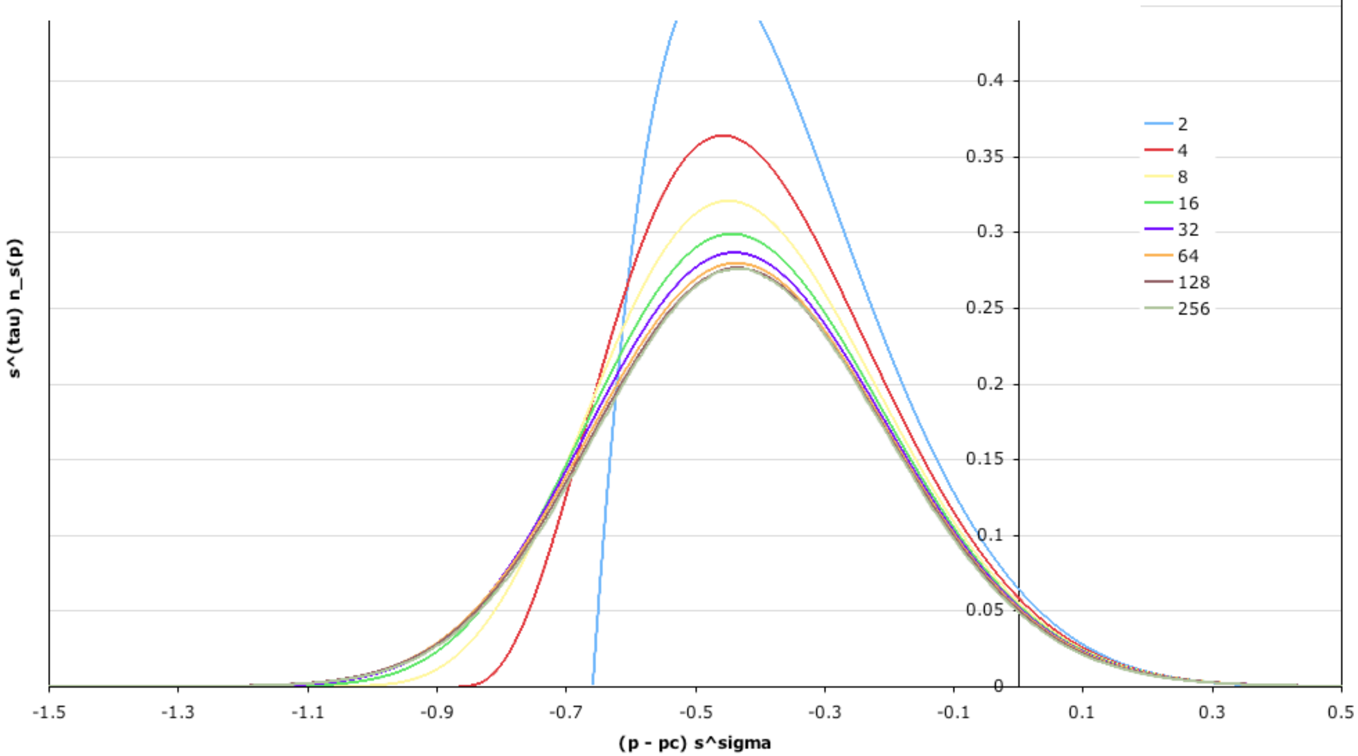
At the critical point,

$n_s(p_c) \sim A\, s^{-\tau}$

$$n_s(p) = As^{-\tau} f(B(p - p_c)s^{\sigma})$$

To test scaling, you can plot

$$s^{\tau} n_s(p) \;\; vs. \;\; (p - p_c)s^{\sigma}$$



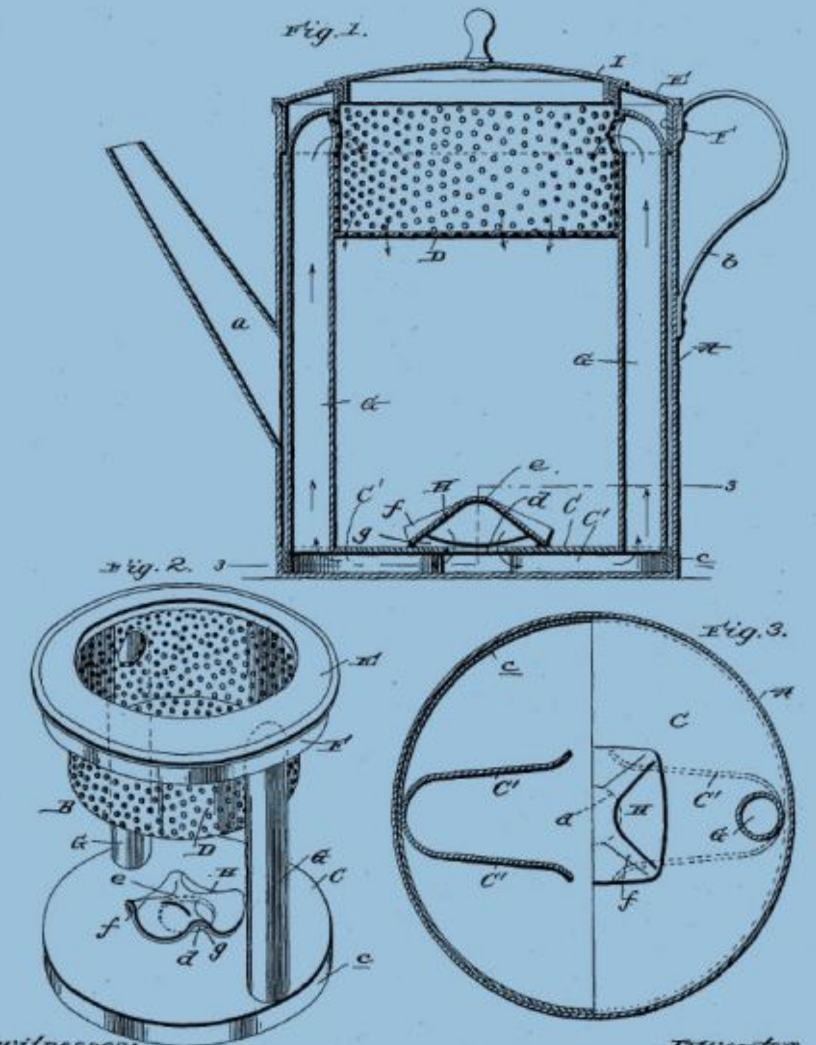
No. 628,705. Patented July 11, 1899.

D. P. GOLDSMITH & B. F. MARTYN.

COFFEE POT.

(Application filed Mar. 14, 1899.)

(No Model.)



Witnesses:
C. H. Raeder
J. G. Carney

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B. F. Martyn
By James J. Sheehy
Attorney