2021-01-28

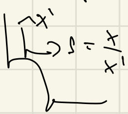
$P \in \mathrm{Diff}_b^*(X)$, $P = \sum_{k=0}^{m} A_k(x)\,(x\partial_x)^k$

$A_k(x) \in \mathrm{Diff}^{m-k}(\partial X)$

$$I_P = \sum_0^m A_k(0)\,(x\partial_x)^k$$

$$\hat I_P(z) = \sum_0^m A_k(0)\, z^k$$

$I_P \;\hat{=}\; \kappa_P' = \beta^*\kappa_P$ restricted to ff.



x', $s = \frac{x}{x'}$

$\kappa_P(s, 0)$ (leave out y, y')

$\overset{\wedge}{x'=0}$ $=: \kappa_{P,ff}(s)$

$I_P u = \kappa_{P,ff} * u$ defines I_P for $P \in \Psi_b^*$

$$\hat I_P(z) = (M\kappa_{P,ff})(-z)$$

Lemma: $\hat I_P(z) = (x^{-z} P x^z)_\partial$, $P \in \Psi_b^*$

Ex: $P = (x\partial_x)^k \Rightarrow x^{-z}(x\partial_x)^k x^z$

$= (x\partial_x + z)^k$

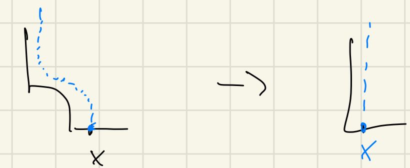
at $x=0$ $\rightsquigarrow$ $= z^k = \hat I_P(z)$.

Proof: Find compute P_∂ in terms of Schwartz kernels.

$v \in C^\infty(\partial X)$

$$(P\tilde v)(x) = \int_0^\infty \overbrace{\kappa_P\left(\frac{x}{x'}, x'\right)}^{\kappa_P(x,x')} \tilde v(x')\,\frac{dx'}{x'}$$

Need: $x \to 0$. A case of SAL.



Change variables to $s = \frac{x}{x'}$, so $x' = \frac{x}{s}$:

$$(P\tilde v)(x) = \int_0^\infty \underbrace{\kappa_P\left(s, \frac{x}{s}\right)}_{\text{rapid decay as } s\to 0 \text{, unif in } x} \tilde v\left(\frac{x}{s}\right)\frac{ds}{s}$$

supported in $\frac{x}{s} < 1$

$$= \int_0^\infty \left[\kappa_P(s,0)\cdot \tilde v(0) + \frac{x}{s}\, O(s^\infty)\right]\frac{ds}{s}$$

$$\xrightarrow{x\to 0} \int_0^\infty \kappa_P(s,0)\,\frac{ds}{s} \cdot \underbrace{\tilde v(0)}_{v}$$

Result:

$$P_\partial v = \left[\int_0^\infty \kappa_{P,ff}(s)\,\frac{ds}{s}\right] v$$
(acting in y)

$$\kappa_{x^{-z} P x^{z}} = \left(\frac{x'}{x}\right)^{z} \cdot \kappa_P = s^{-z}\cdot \kappa_P$$

$$\Rightarrow (x^{-z} P x^{z})_\partial = \int_0^\infty \kappa_{P,ff}(s)\, s^{-z}\,\frac{ds}{s}$$

$$= (\mathcal{M}\kappa_{P,ff})(-z) \qquad \text{q.e.d.}$$

Thm: a) Let $P \in \Psi_b^m$, $Q \in \Psi_b^\ell$. Then

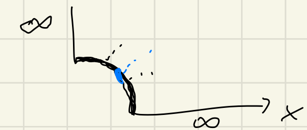
$$\hat{I}_{PQ}(z) = \hat{I}_P(z)\,\hat{I}_Q(z)$$
(in $\Psi^m(\partial X)$)

$$I_{PQ} = I_P\, I_Q \quad (\text{in } \Psi^m_{b,I}(X))$$

b) there is a short exact sequence

$$0 \to x\,\Psi_b^m(X) \to \Psi_b^m(X) \xrightarrow{I} \Psi^m_{b,I}(\tilde{X}) \to 0$$

(b):



Rem: $x\cdot \Psi_b^m = \{\kappa_P' \in \Psi_b^m : \kappa'_{P|ff} = 0\}$

$(= \rho_{ff}\cdot \Psi_b^m)$

Proof of a): for $\hat{I}_P$: First, $(PQ)_\partial = P_\partial Q_\partial$

(Proof: $v \in C^\infty(\partial X)$, extend to $\tilde{v}$

$w = Q_\partial v$, extended by $\tilde{w} = Q\tilde{v}$

$\Rightarrow P_\partial Q_\partial v = P_\partial w = (P\tilde{w})|_{\partial X}$

$= (PQ\tilde{v})|_{\partial X} = (PQ)_\partial v.$)

Next: $x^{-z}PQx^{z} = x^{-z}Px^{z}\; x^{-z}Qx^{z}$

$\Rightarrow$ ✓.

• For I_P: Recall $\hat{I}_P = \mathcal{M}_- I_P$

$$I_P u = \mathcal{M}^{-1}(\hat{I}_P) \dot{\ast} u$$

$$\begin{aligned}
\Rightarrow \quad I_{PQ} u &= \mathcal{M}^{-1}(\hat{I}_{PQ}) \dot{\ast} u \\
&= \mathcal{M}^{-1}(\hat{I}_P \hat{I}_Q) \dot{\ast} u \\
&= \left[\mathcal{M}^{-1}(\hat{I}_P) \dot{\ast} \mathcal{M}^{-1}(\hat{I}_Q)\right] \dot{\ast} u \\
&= \dots \quad [\qquad] \\
&= I_P (I_Q u)
\end{aligned}$$

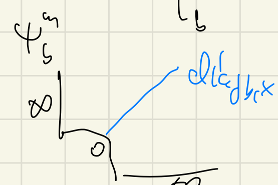
$$\int K(x,x') u(x') \frac{dx'}{x'}$$

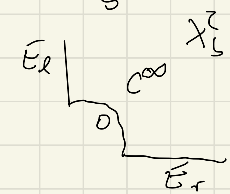
4.2.2 Definition of full b-calculus
mapping + composition theorems

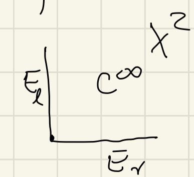
<u>Def</u>: For $\mathcal{E} = (E_l, E_r)$ let

$$\Psi_b^{m,\mathcal{E}}(X) := \Psi_b^m(X) + \Psi_b^{-\infty,\mathcal{E}}(X)$$

where $\Psi_b^{-\infty,\mathcal{E}} := \tilde{\Psi}_b^{-\infty,\mathcal{E}} + \Psi^{-\infty,\mathcal{E}}$

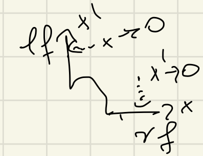






Think of $r \mathrel{\hat{=}}$ input $(x' \to 0)$

$l \mathrel{\hat{=}}$ output $(x \to 0)$



<u>Mapping thm</u>:

$P \in \Psi_b^{m,\mathcal{E}}$, $u \in \mathcal{A}^F$, then:

If $E_r + F > 0$ then Pu is defined and

$$Pu \in \mathcal{A}^{E_l \bar{\cup} F}$$

Composition: $P \in \Psi_b^{m,\mathcal{E}}$, $Q \in \Psi_b^{\ell,\mathcal{F}}$

If $E_r + F_\ell > 0$ then PQ is defined and

$$PQ \in \Psi_b^{m+\ell,\mathcal{G}}$$

where $G_\ell = E_\ell \bar\cup F_\ell$, $G_r = E_r \bar\cup F_r$

(Pf. as before) Note: output asymp. involves output of both E, F. This arises from b-blow-up!

4.2.3 Inverting the indicial operator

Steps:

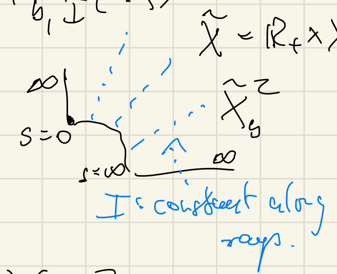
- invert the indicial family $\rightsquigarrow$ meromorphic family of ops on ∂X
- investigate Mellin tf. ptg $\Leftrightarrow$ meromorphic
- get $(I_P)^{-1}$ by inverse Mellin tf. from $(\hat{I}_P)^{-1}$.

Rem: We need an inverse of I_P (parametrix of I_P would not be enough)

Notation in this section:

Start with $\tilde P \in \Psi^m_{b,I}(\tilde X)$

(Later: $\tilde P = I_P$).



Also: $\hat P(z) = \mathcal{M}(\kappa_{\tilde P})(-z)$

$\kappa_{\tilde P}(s)$

Note: $\hat P$ entire (holomorphic on $\mathbb{C}$)

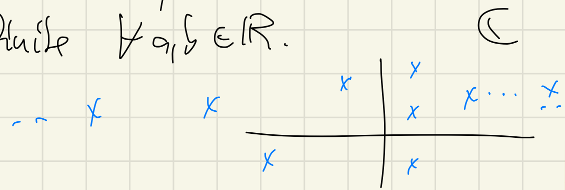
Inverting $\hat P(z)$

Def: $\mathrm{spec}_b(\tilde P) = \{ z \in \mathbb{C} :$ $\hat P(z) : C^\infty(\partial X) \circlearrowleft$ is not invertible $\}$

If $\tilde P = I_P$, then $\mathrm{spec}_b(P) := \mathrm{spec}_b(\tilde P)$.

Prop: Assume $\tilde P$ is elliptic.

a) $\operatorname{spec}_b(\tilde P)$ is „vertically finite", i.e. $\operatorname{spec}_b(\tilde P) \cap \{a \le \operatorname{Re} z \le b\}$ is finite $\forall a, b \in \mathbb{R}$.



b) $\hat P(z)^{-1}$ exists for $z \notin \operatorname{spec}_b(\tilde P)$ and is meromorphic with finite rank poles:

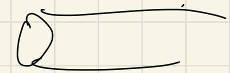
For each $z_0 \in \operatorname{spec}_b(\tilde P)$, for z near z_0:

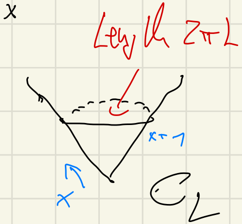
$$\hat P(z)^{-1} = \sum_{\ell=-N}^{-1} \frac{B_\ell}{(z-z_0)^\ell} + \left(\text{holom. in } z \text{ at } z = z_0\right)$$

and all B_ℓ have finite rank.

(also: ran $B_\ell \subset C^\infty(\partial X)$).

Example: $X = \tilde X = S^1_y \times \mathbb{R}_{+,x}$





$$P = \tilde P = (x\partial_x)^2 + L^{-2}\partial_y^2.$$

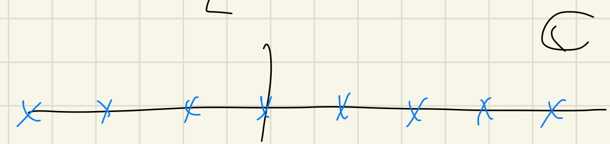
(if $L=1$: $C_L = \mathbb{R}^2$)

$$\hat P(z) = z^2 + L^{-2}\partial_y^2 \in \operatorname{Diff}^2(S^1)$$

Invertible iff $z^2 \notin \operatorname{spec}(-L^{-2}\partial_y^2)$.

$\operatorname{spec}(-\partial_y^2) = \{k^2 : k \in \mathbb{Z}\}$. eigenfcns e^{iky}.

$\rightsquigarrow \operatorname{spec}_b(P) = \frac{1}{L} \cdot \mathbb{Z}$



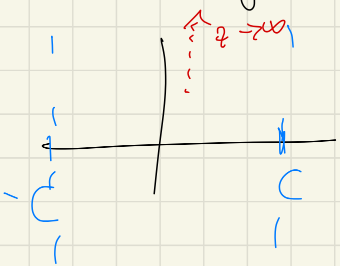
Proof of Proposition:

1. Do it for $I + \tilde{R}$, $\tilde{R} \in \Psi_{b,I}^{-\infty}$
   - invertible for "large" z
   - analytic Fredholm theorem
2. Use small parametrix to reduce general $\tilde{P}$ to this case.

Lemma: $\tilde{R} \in \Psi_{b,I}^{-\infty} \Rightarrow \hat{R}(z) \in \Psi^{-\infty}(\partial X)$ is entire and $\forall N, C$:

$$\| \hat{R}(z) \|_{L^2 \to L^2} = O(\langle z \rangle^{-N})$$

uniformly for $|\mathrm{Re}\, z| \le C$.



$\mathbb{C}$

$\langle z \rangle := 1 + |z|$

Pf: $K_{\tilde{R}}$ is smooth and rapidly decaying as $s \to 0$, $s \to \infty$.

$$\hat{R}(z)(y,y') = \int_0^\infty K_{\tilde{R}}(s,y,y') s^{-z} \frac{ds}{s} \qquad (*)$$

Use $\|\hat{R}(z)\|_{L^2 \to L^2} \le \iint_{\partial X\, \partial X} |\hat{R}(z)(y,y')|^2 \, dy\, dy'$

For $|\mathrm{Re}\, z| \le C$ this is bounded:

$s < 1$: Use $K = O(s^{C+1})$ $(s \to 0)$

$s > 1$: — " — s^{-C-1} $(s \to \infty)$

Also, easily $s^{-z} = (-z)^{-N} (s\partial_s)^N s^{-z}$

and int. by parts we get that

$|z|^N$ times $(*)$ is also bounded.

qed

Analytic Fredholm theorem

Let $z \mapsto A(z)$ be a holomorphic (on $\mathbb{C}$) family of compact operators on a Hilbert space $\mathcal{H}$.

If $I + A(z)$ is invertible for some z then $(I+A(z))^{-1}$ is meromorphic with finite rank poles.

Proof of Proposition: $\hat{P} \in \Psi^{m}_{b,I}(\hat{X})$ elliptic.

Let $Q \in \Psi_b^{-m}(\hat{X})$ be a small parametrix:

$$\hat{P} Q = Id + R, \quad R \in \Psi_b^{-\infty}.$$

Take $\hat{I}$:

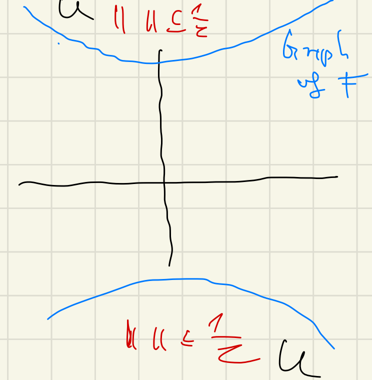
$$\hat{P}(z)\, I_Q(z) = Id + \underbrace{I_R(z)}_{\hat{R}(z)}$$

By the lemma:

$$\|\hat{R}(z)\|_{L^2 \to L^2} \le \frac{1}{2} \text{ on some set}$$

$$U = \{ z : |\operatorname{Im} z| > F(|\operatorname{Re} z|) \}$$

for some function F



$\Rightarrow Id + \hat{R}(z)$ is invertible for $z \in U$

The inverse has the form

$$Id + \hat{S}(z), \quad \hat{S}(z) \in \Psi^{-\infty}(\partial X)$$

[Since Id is parametrix for $Id + \hat{R}(z)$]

AFT (for $A = \hat{R}$)

$\Rightarrow \hat{S}(z)$ is meromorphic, poles in $\mathbb{C} \setminus U$.

$$\Rightarrow \hat{P}(z) \text{ has inverse } \underbrace{I_Q(z)}_{\text{holom.}} \big(\underbrace{Id + S(z)}_{\text{merom.}}\big)$$

q.e.d.