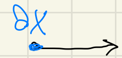
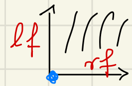
2020-12-10

b-calculus (smoothing part)

X manifold with boundary, eg $X = \mathbb{R}_+$

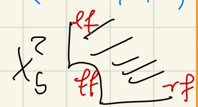


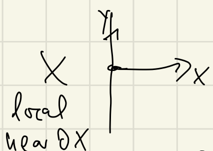
$X^2 = X \times X$ mfd with corners, $(\partial X)^2$ corner of codim. 2

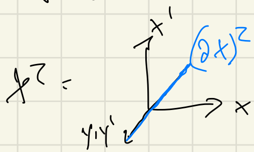


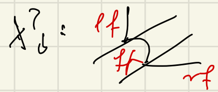
Def: $X^2_b := [X^2, (\partial X)^2]$ $\qquad (\partial X)^2 = \{(0,0)\}$

b-double space of X.









coord's on X^2: $(x, y; x', y')$

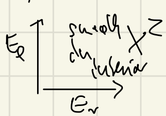
Notation: $\quad lf = (\partial X) \times X$ $\qquad$ resp. their lifts to X^2_b

$\qquad\qquad rf = X \times \partial X$

$\qquad\qquad ff = \beta^*((\partial X)^2)$

Def: Let $\mathcal{E} = (E_l, E_r)$ be a pair of index sets.

a) $\Psi^{-\infty,\mathcal{E}}(X) := \{$operators on X with Schwartz kernels in $\mathcal{A}^{(E_l,E_r)}(X^2, |\Omega|^{1/2})\}$



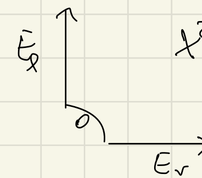
operators on X = operators on $C_0^\infty(X, |\Omega_b|^{1/2})$

$\longleftrightarrow$ Schwartz kernels

b) $\tilde{\Psi}_b^{-\infty,\mathcal{E}}(X) := \{$op's on X with S. kernels K so that $\beta^* K \in \mathcal{A}^{(E_l, 0, E_r)}(X^2_b)\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ lf $\quad$ ff $\quad$ rf

$\beta: X^2_b \to X^2$.

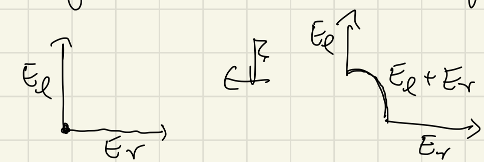


$0 = \mathbb{N}_0 \times \{0\}$ ≙ smooth up to ff.

c) $\Psi_b^{-\infty,\mathcal{E}}(X) := \Psi^{-\infty,\mathcal{E}}(X) + \tilde{\Psi}_b^{-\infty,\mathcal{E}}(X)$.

"smoothing b-calculus with index family $\mathcal{E}$".

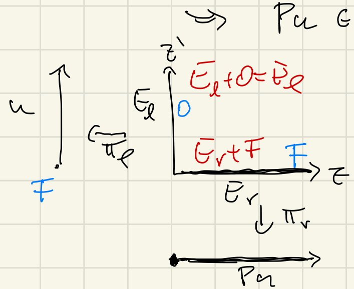
Rem:



$\rightsquigarrow \Psi^{-\infty,\mathcal{E}} \not\subset \tilde{\Psi}_b^{-\infty,\mathcal{E}}$

Mapping properties: $u \in \mathcal{A}^F(X)$, $P \in \Psi^{-\infty,\mathcal{E}}(X)$

$\Rightarrow \quad Pu \in \; ?$



$$(Pu)(z) = \int K(z,z')\,u(z')\,dz'$$

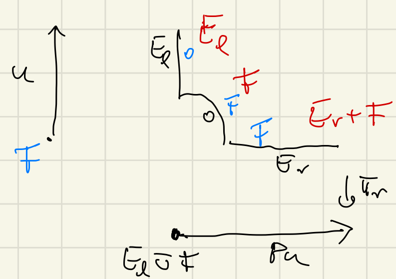
$$Pu = \pi_{r*}\left(K \cdot \pi_l^* u\right)$$

(Int.): $\inf(E_r + F) > 0$

Then: $\quad Pu \in \mathcal{A}^{E_l}(X)$

(PFT)

Now: $\quad P \in \Psi_b^{-\infty,\mathcal{E}}(X)$:



int. condition:

$$\inf(E_r + F) > 0$$

PFT:

Result:

$$Pu \in \mathcal{A}^{E_l \bar{\cup} F}(X).$$

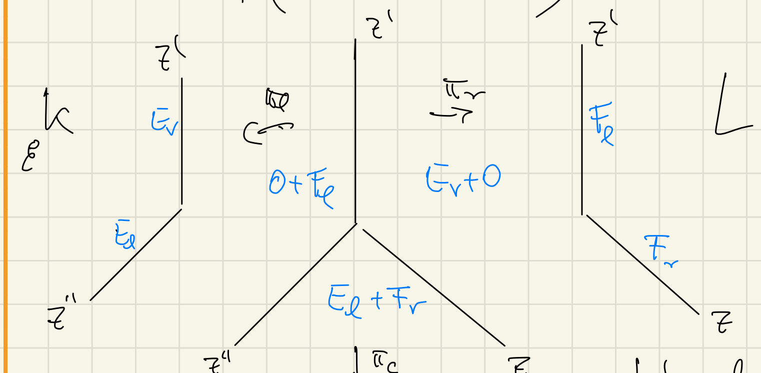
Composition $\quad \Psi^{-\infty,\mathcal{E}}$

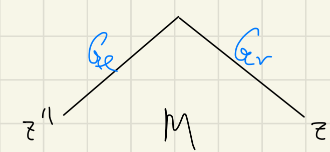
$$P_M = P_L \circ P_K$$

$$M(z,z'') = \int L(z,z')\,K(z',z'')\,dz'$$

$$\begin{tikzcd} & X^3 := X \times X \times X \ni (z,z',z'') \arrow[dl, "\pi_l"'] \arrow[d, "\pi_c"] \arrow[dr, "\pi_r"] & \\ (z',z'') & (z,z'') & (z,z') \end{tikzcd}$$

$$M = \pi_{c*}\left(\pi_r^* L \cdot \pi_l^* K\right)$$

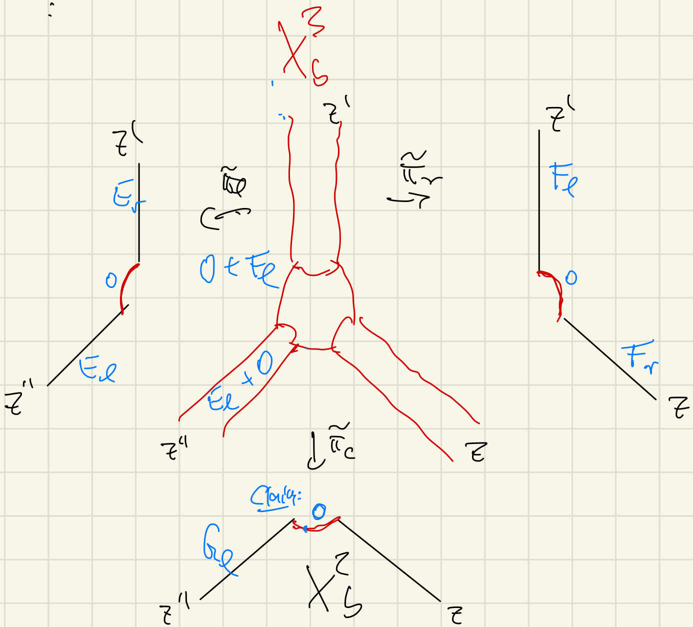




Int. cond.:
$\inf(E_l + F_r) > 0$.

Result:
$G_l = F_l$, $G_r = E_r$.

$\tilde{\Psi}_b^{-\infty,\mathcal{E}}$:



Blow-up triple space: X^3 :

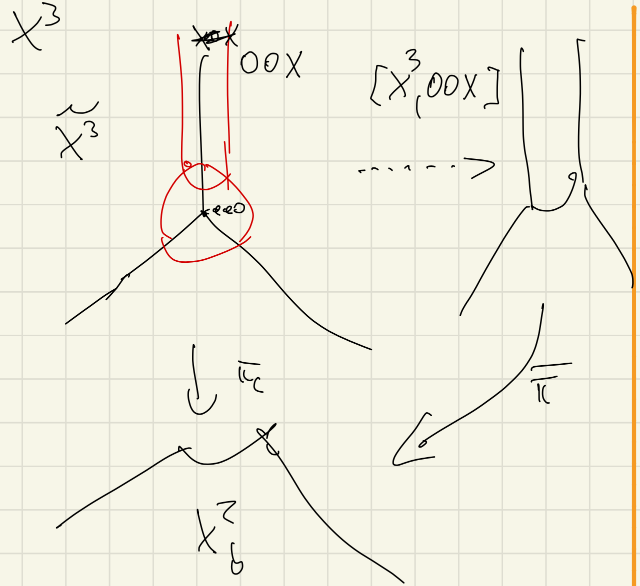
$$X^3_b := \left[X^3 ; (\partial X)^3, (\partial X)^2 \times X, (\partial X) \times X \times (\partial X), X \times (\partial X)^2 \right]$$

b-triple space.

<u>Lemma</u>: The lifted projections are b-fibrations.
$$X^3_b \to X^2_b$$

<u>Thm</u>: Let $K \in \Psi_b^{-\infty,\mathcal{E}}$, $L \in \Psi_b^{-\infty,\mathcal{F}}$.
Then $KL \in \Psi_b^{-\infty,\mathcal{G}}$ if $\inf(E_\ell + F_r) > 0$,
and $G_\ell = E_\ell \bar{\cup} F_\ell$
$G_r = E_r \bar{\cup} F_r$

<u>Proof</u>: See picture for $\tilde{\Psi}_b \circ \tilde{\Psi}_b$ for G_ℓ.
(result: at lf + rf, $\tilde{\Psi}_b \circ \tilde{\Psi}_b \to \tilde{\Psi}_b$)
<u>But</u>: at ff: $\tilde{\Psi}_b \circ \tilde{\Psi}_b$ <u>not</u> smooth at ff.
But: $\tilde{\Psi}_b \circ \tilde{\Psi}_b \subset \tilde{\Psi}_b + \Psi$
(see Melrose: APS-book, p. 200)



$X = \mathbb{R}_+$: Claim: π_c lifts to a map $\tilde{\pi}_c : X^3_b \to X^3_b$.

$$X^3_b = [X^3; 000, 0X0, 00X, X00] \xrightarrow{!} [X^2, 00]$$

(disjoint)

$$\uparrow \bar{\pi}$$

blow-down for $0X0$, $X00$ $\downarrow$

$[X^3; 000, 00X]$ $(= \widetilde{X}^3)$

$\|$ ← because $000 \subset 00X$ + commutation theorem

$[X^3; 00X, 000]$ → $[X^3, 00X] = [X^2, 00] \times X$

blow-down 000

3.4 Conormal distributions and classical ΨDO calculus

ΨDO = pseudodifferential operator.

Will see: Singular ΨDO calculus

= classical ΨDO calculus
+ smoothing op's with phg expansions (as above)

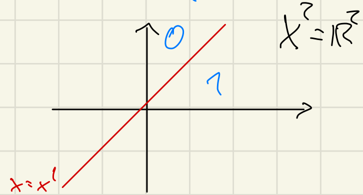
Problem: Sol'n operators of PDE are integral operators, but the kernels are not smooth in the interior of X^2

ex: • $u' = f$ on $\mathbb{R}$. $f \in C_0^\infty$

$$u(x) = \int_{-\infty}^{x} f(x')\,dx' = \int_{-\infty}^{\infty} \underbrace{H(x-x')}_{k(x,x')} f(x')\,dx'$$

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Heaviside function.



• $\Delta u = f$ in $\mathbb{R}^3$

$f \in C_0^\infty$

$$u(z) = \int_{\mathbb{R}^3} \underbrace{\frac{c}{|z-z'|}}_{k(z,z')} f(z')\,dz'$$

These are (non-smooth) functions, but it is better to think of them as distributions.

Plan:
- review distribution language
- conormal distributions
- ΨDO = Op's whose Schwartz kernels are conormal distr. (w.r.to diagonal)