2020-12-09

3.3 Push forward and pull-back theorems

Ex: $f(x,y) = x\cdot y$ $\quad \mathbb{R}^2_+ \to \mathbb{R}_+$.

$\mu \qquad f_*\mu$

$f: X \to Y$ b-map, interior

bhs $\quad G \quad H \qquad e(G,H)\in \mathbb{N}_0$

$G \underset{f}{\sim} H :\iff e(G,H) > 0$.

$\bar f : \mathcal{M}(X) \to \mathcal{M}(Y)$

$F \mapsto F'$ if $\forall p \in F^\circ : f(p) \in (F')^\circ$

Def: f is b-fibration if it is surjective and

a) $\forall G \in \mathcal{M}_1(X)$: codim $\bar f(G) \le 1$

b) $f|_{F^\circ} \to \bar f(F)^\circ$ is a fibration $\forall F \in \mathcal{M}(X)$.

Lemma: f is a b-fibration if and only if

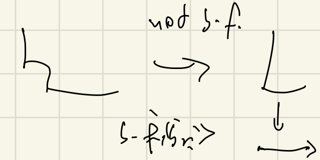
(i) $f_* : {}^bT_pX \to {}^bT_{f(p)}X$ is surjective $\forall p \in X$

(f is b-submersion)

(ii) $f_* : {}^bN_p \to {}^bN_{f(p)}$ is surjective $\forall p \in X$.

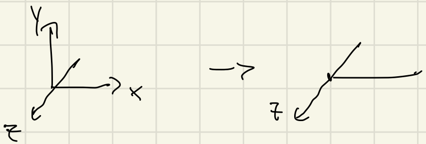
(f is b-normal)

Ex: •



• $Y = \mathbb{R}_+$: f is b-fibration iff surjective and $f: X^\circ \to \mathbb{R}_+^\circ$ is fibration.

•



$(x,y,z) \mapsto (xy, z)$ is b-fibr.

Rem: (a) $\iff \forall G$ there is at most one H so that $G \underset{f}{\sim} H$

Notation: E index set. $\inf E := \{\inf \operatorname{Re} z : (z,k) \in E\}$

$c \cdot E := \{(cz,k) : (z,k) \in E\}$

Push-forward theorem: $f: X \to Y$ b-fibration.

$\mathcal{E}$ index family for X.

Assume: $\inf \mathcal{E}(G) > 0$ if $\bar f(G) \to Y$ (Int)

then: $\mu \in \mathcal{A}^{\mathcal{E}}(X, |\Omega_b|)$, f proper on supp μ

$\implies f_*\mu \in \mathcal{A}^{f_\# \mathcal{E}}(Y, |\Omega_b|)$

$(f_\#\mathcal{E})(H) = \overline{\bigcup_{G \underset{f}{\sim} H}} \frac{1}{e(G,H)} \mathcal{E}(G)$

Push-forward theorem: $f: X \to Y$ b-fibration.

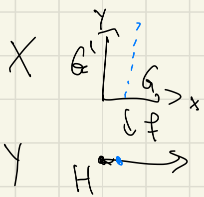
$\mathcal{E}$ index family for X.

Assume: $\inf \mathcal{E}(G) > 0$ if $f(G) \to Y$ (Int)

then: $\mu \in \mathcal{A}^{\mathcal{E}}(X, |\Omega_b|)$, f proper on supp μ

$\Rightarrow$ $f_*\mu \in \mathcal{A}^{f_\#\mathcal{E}}(Y, |\Omega_b|)$

$$(f_\#\mathcal{E})(H) = \overline{\bigcup_{G \xrightarrow{f} H}} \frac{1}{e(G,H)} \mathcal{E}(G)$$

Ex: 

Int: $\inf \mathcal{E}(G) > 0$.

$$\int_0^1 |x^z \log^k x| \cdot \frac{dx}{x} < \infty$$

$\iff \operatorname{Re} z > 0$.

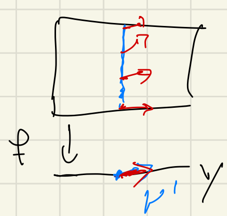
Main points of proof:

· Conormal regularity: $\mu \in \mathcal{A}^s \Rightarrow f_*\mu \in \mathcal{A}^{s'}$

$f_*: {}^bT \to {}^bT$ surjective $\Rightarrow$

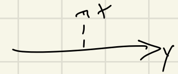
Any $V' \in \mathcal{V}_b(Y)$ can be lifted to a $V \in \mathcal{V}_b(X)$:

$$f_* V = V'.$$

(Compare: f ~~fibration~~ submersion 

)

Then $V' f_*\mu = f_*(V\mu)$ (apply several times)

(model: $V' = a x\partial_x + b\partial_y$ 

)

· Polyhomogeneous regularity:

$V_G \in \mathcal{V}_b(X)$ is normal for G if

$$V_G|_G = x\partial_x, \quad G = \{x = 0\}.$$

Phg defined in terms of applying $(V_G - z)$ to μ.

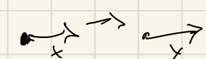
(e.g. $x\partial_x - z$)

f b-normal: ${}^bN \to {}^bN$ surjective $\Rightarrow$

For $H \in M_1(Y)$, $V' = V'_H$ normal for H

then for each $G \xrightarrow{f} H$ $\exists$ V_G normal for G s.t.

$$f_* V_G = e(G,H)\, V_H$$

(ex: $f(x) = y = x^2$: $f_*(x\partial_x) = x \cdot \frac{\partial y}{\partial x}\partial_y = 2y\partial_y$ 

)

Then $\underbrace{(e\cdot V'_H - z)}_{e\cdot(V'_H - \frac{z}{e})} f_*\mu = f_*[(V_G - z)\mu]$

$e = e(G,H)$

qed

3.3.4 Pull-back theorem

Thm: $f: X \to Y$ interior b-map, $v \in \mathcal{A}^{\mathcal{F}}(Y)$.

then $\qquad f^* v \in \mathcal{A}^{f^\# \mathcal{F}}(X)$

where
$$(f^\# \mathcal{F})(G) = \sum_{G \underset{f}{\sim} H} e(G,H) \cdot \mathcal{F}(H)$$

$$:= \Big\{ \big(\ell + \sum_{G \underset{f}{\sim} H} e(G,H) \cdot z_H \,,\ \sum_{G \underset{f}{\sim} H} k_H \big) :$$
$$(z_H, k_H) \in \mathcal{F}(H),\ \ \ell \in \mathbb{N}_0 \Big\}$$

Main points of proof:

- $f(x,y) = x^a y^b = t$.
$$f^*(t^z \log^k t) = x^{az} y^{bz} \cdot (a \log x + b \log y)^k$$
$$= \text{sum of } \ x^z \log^\ell x \cdot y^z \log^{k-\ell} y$$

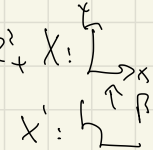
- Several t-variables:
$$f^*(t_1^{z_1} t_2^{z_2}) \leadsto \text{sum of exponents.}$$
$t_1 = x$

Recall resolutions of functions.

Def $\beta: X' \to X$ iterated blow-down map.

β resolves $v: X^\circ \to \mathbb{C}$ $:\Leftrightarrow$ $\beta^* v$ is phg.

Ex: $X = \mathbb{R}^2_+$ $\qquad f(x,y) = \sqrt{x^2+y^2}$



X: y, x; X': β

β resolves f.

Rem: More blow-ups keep v resolved.

Rem: Pull-back densities under blow-down maps:

$\beta: [X, Y] \to X$ $\qquad$ ($Y \subset X$ p-submfd.)

μ density

$\leadsto$ On $[X,Y] \setminus ff$ have $\beta^* \mu := (\beta^{-1})_* \mu$

$\downarrow \subset \cdots$ diffeo.

$X \setminus Y$

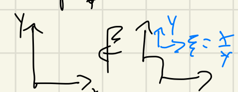
Lemma: a) $\mu \in C^\infty(X, |\Omega_b|) \Rightarrow \beta^* \mu \in C^\infty([X,Y], |\Omega_b|)$

b) If Y is a face of X then:

$X' = [X,Y]$ $\qquad \mu$ non-vanishing $\Rightarrow \beta^* \mu$ non-vanishing.

Proof: $\beta_*: {}^bTX' \to {}^bTX \Rightarrow \beta^*: {}^bT^*X \to {}^bT^*X'$.

Ex:

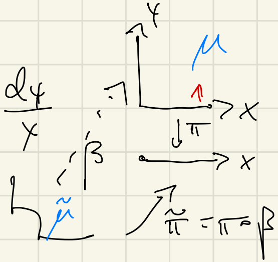


$\beta^*\left(\frac{d\xi}{\xi} \frac{dy}{y} \right) = \frac{d(x/y)}{x/y} \frac{dy}{y} = \frac{dx}{x} \frac{dy}{y}$

An example for using PFT:

What is the regularity of $v(x) = \int_0^1 \sqrt{x^2+y^2}\,dy$ at $x=0$?

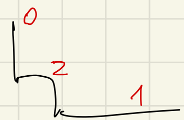
- Translate: $v\frac{dx}{x} = \pi_* \mu$, $\mu = y\sqrt{x^2+y^2}\,\frac{dx}{x}\,\frac{dy}{y}$



- Resolve: $\tilde\mu = \beta^*\mu$, $\tilde\pi = \pi\circ\beta$

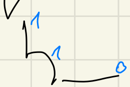
  $\Rightarrow \pi_*\mu = \tilde\pi_*\tilde\mu$

- Apply PFT:
  - $\tilde\pi$ b-fibration
  - $\tilde\mu$ phg b-density, index sets



(notation: $a := (a+\mathbb{N}_0)\times\{0\}$)

  - integrability condition: $1 > 0$ ✓
  - exponents $e(G,H)$: $H=\{0\}$:

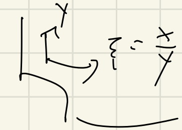


PFT: $v \in \mathcal{A}^{\tilde\pi_\# \mathcal{E}}(\mathbb{R}_+)$

$t = x$ $\quad \tilde\pi_\#\mathcal{E}(H) = 0 \,\bar\cup\, 2 = \{(0,0),(1,0)\} \cup \left((2+\mathbb{N}_0)\times\{0,1\}\right)$

$t^0, t^1, t^2, t^2\log t, t^3, t^3\log t \dots$

More precise:



$$\tilde\mu = y^2\sqrt{\xi^2+1}\,\frac{d\xi}{\xi}\,\frac{dy}{y}$$

and $\tilde\pi(\xi, y) = y\cdot\xi$

Log-terms appear at diagonal terms of Taylor series at $y=\xi=0$:

$$y^2\cdot\left(1+\frac{\xi^2}{2}-\frac{\xi^4}{8}\dots\right)$$

$\rightsquigarrow$ only one log term: $-\frac{1}{2}x^2\log x$

3.3.5 Schwartz kernels and half-densities

X manifold (with corners). Consider integral operator:

$$(P_K u)(z) = \int_X K(z,z')\, u(z')\, dz' \qquad z, z' \in X$$

K function on $X \times X$ (later: distribution)

Need to choose measure dz' on X.

Better not to have it, better include it in u or K.

Options:

K	acts on	result is
function	$u\,dz'$ density	function
$K\,dz'$	u function	function
$K\,dz\,dz'$	u function	density

no composition possible

fcn. dz · density in z'

Solution: $K\sqrt{dz\cdot dz'}$ $\quad u\cdot\sqrt{dz'}$ $\quad (Pu)\cdot\sqrt{dz}$

All same kind of object!

Recall: density is locally $v(z)\,dz$

Def: A half-density on a manifold is an object which in local coords is

$$v(z)\sqrt{dz}$$

meaning that: $v(z)\sqrt{dz} = w(\tilde z)\sqrt{d\tilde z}$

$$\rightsquigarrow \quad w = v\cdot\sqrt{\left|\det \frac{\partial z_i}{\partial \tilde z_j}\right|}$$

Rem: • α, β half-densities (smooth, cpct support say)

$\Rightarrow \int_X \alpha\cdot\beta$ makes sense

So $L^2(X, |\Omega|^{1/2})$ is defined without choice of measure.

• $C^\infty(X, |\Omega|^{1/2})$, $C^\infty(X, |\Omega_b|^{1/2})$, $\mathcal{A}^{\mathcal{E}}(X, |\Omega_b|^{1/2})$ defined in analogous ways as for densities.

$$\sqrt{\frac{dx_1}{x_1}\cdots\frac{dx_k}{x_k}\, dy_1 \cdots dy_{n-k}}\,.$$