2020-11-18

the b-tangent bundle

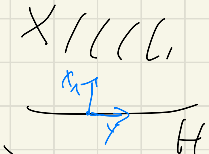
X manifold with corners. $(x_1 \cdots x_k\ y_1 \cdots y_{n-k})$

$TX \to X$ tangent bundle $\quad \partial_{x_i}, \partial_{y_j}$

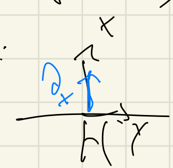
${}^bTX \to X$ b-vectorbundle $\quad x_i\partial_{x_i}, \partial_{y_j}$

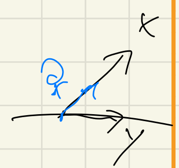
Sections of ${}^bTX$ = b-vector fields

Lemma: Let H be a bound. hypersurface of X. Let $H = \{x_i = 0\}$ locally, for coords (x,y). Then $x\partial_x$ is indep. of the choice of coords. (as elt of ${}^bTX$)



Rem: False for $\partial_x \in TX$:





Proof: $\tilde x, \tilde y$ other coord. system, $H = \{\tilde x = 0\}$

$\Rightarrow x = a\cdot\tilde x$, $a > 0$ smooth function

$$\tilde x \frac{\partial}{\partial \tilde x} = \frac{x}{a}\cdot\left(\frac{\partial x}{\partial\tilde x}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \tilde x}\frac{\partial}{\partial y}\right)$$

$$= \frac{x}{a}\cdot\left((a+\tilde x\cdot a_{\tilde x})\partial_x + b\cdot\partial_y\right)$$

$$= x\partial_x + \underbrace{O(x^2)\partial_x + O(x)\partial_y}_{=0 \text{ at } x=0, \text{ as element of } {}^bTX.}$$

qed

Rem: • $x\partial_x$ is a natural section of ${}^bT_HX$ (x a bd. fcn. of H)

• $T_pH \subset T_pX$, $p\in H$, naturally.

but: ${}^bT_pH \not\subset {}^bT_pX$. (naturally: ${}^bT_pH \cong {}^bT_pX / \mathrm{span}\{x\partial_x\}$)

Anchor map:

Any $V \in {}^b\mathcal{V}(X)$ is also in $\mathcal{V}(X)$

$\rightsquigarrow \quad {}^b\mathcal{V}(X) \longrightarrow \mathcal{V}(X)$

This induces a linear map

$$\iota_p : {}^bT_pX \longrightarrow T_pX \qquad \forall p \in X$$

Note: $\iota_p(x\partial_x) = 0$ at $x = 0$.

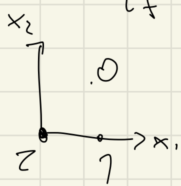
- ι_p is isomorphism iff $p \in X^\circ$

Def: b-normal space at p is

$$\ker \iota_p = \operatorname{span}\{x_1\partial_{x_1}, \dots, x_k\partial_{x_k}\}$$

if $p \in \{x_1 = \dots = x_k = 0\}$.

$X = \mathbb{R}^2_+$:



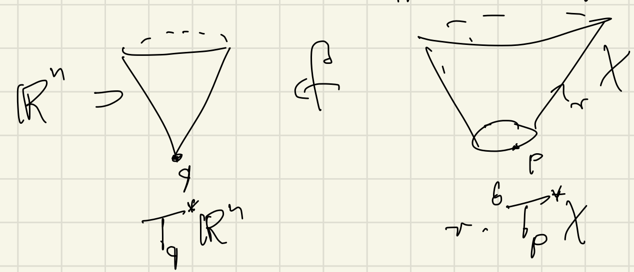
Dual:

$${}^bT^*X := ({}^bTX)^*$$

$$= \operatorname{span}\left\{ \frac{dx_i}{x_i},\ i = 1 \dots k;\quad dy_j,\ j = 1 \dots n-k \right\}$$

dual basis to $x_i\partial_{x_i}, \partial_{y_j}$.

Rem: ${}^bTX$, ${}^bT^*X$ appear naturally, e.g.:



$$\beta^*(dx_i) = r \cdot (\text{basis of } {}^bT_p^*X)$$

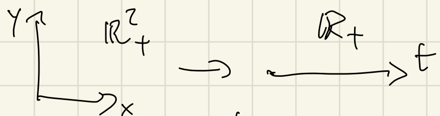
Will see: Only $\frac{dx}{x}, dy$ (rather than dx, dy) very useful, e.g.:

Mellin transform: $\hat{u}(z) = \int_0^\infty u(x)\, x^z \frac{dx}{x}$

$$(x\partial_x)\, x^z = z \cdot x^z$$

b-differential of b-maps

ex:



$f(x,y) = x^a y^b$ $\quad a,b \in \mathbb{N}_0$.

$x\frac{\partial f}{\partial x} = a \cdot f$, $\quad y\frac{\partial f}{\partial y} = b\cdot f$: $\quad df(x\partial_x) = a\cdot t\partial_t$, $\quad df(y\partial_y) = b\, t\partial_t$

Dually: $f^*\left(\frac{dt}{t}\right) = \frac{d(x^a y^b)}{x^a y^b} = \dots = a\cdot \frac{dx}{x} + b\frac{dy}{y}$

Prop: $f: X \to Y$ interior b-map. Then df defines a map

$df: T_p X \to T_{f(p)} Y$

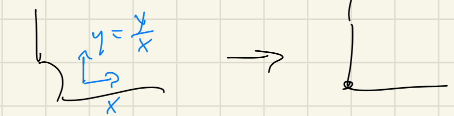
$^b df: {}^bT_p X \to {}^bT_{f(p)} Y$

${}^bN_p X \to {}^bN_{f(p)} Y$

in terms of the basis $\rho_G \partial_{\rho_G}$ (ρ_G bdf of $G \in M_1(X)$) have

$$df(\rho_G \partial_{\rho_G}) = \sum_{H \in M_1(Y)} e(G,H)\, \rho_H \partial_{\rho_H}.$$

Ex: $f = \beta$:



$f(x,y) = (x, xy) \Rightarrow {}^b df$: $\quad x\partial_x \to x\partial_x + y\partial_y$

$\qquad y\partial_y \to y\partial_y$

Blow-up of $(0,0)$ defines a subdivision of ${}^bN_{(0,0)}$ (a fan).

Conversely, fans define (generalized) blow-ups.

(Kottke–Melrose 2015)

Compare: toric varieties from polyhedral fans.

3. Basic notions: Analysis

Recall basic themes:

- local product structure
- b-vector fields, $x\partial_x$

Connected e.g. by blow-up:
- used to create lps
- $x\partial_x$ behave very nicely w.r.t. b-maps, eg blow-down maps.

3.1 Polyhomogeneous (≈-smooth) functions

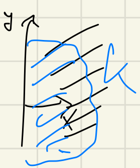
- asymptotics $x^z \log^k x := x^z (\log x)^k$ $\leftrightarrow x\partial_x$
- sums of products at corners
- always smooth in interior.

General notation: Let X mwc, $u,v: X^\circ \to \mathbb{C}$

$u = O(v) \;:\iff \forall K \subset X$ compact $\exists C$:

$|u(p)| \le C \cdot |v(p)| \quad \forall p \in K \cap X^\circ$

(local uniformity)

ex: 

$u(x) = O(x)$

7) "Polyhomogeneous function on X" is function $u: X^\circ \to \mathbb{C}$ with certain behavior near the boundary.

3.1.1. Phg functions on $\mathbb{R}_+$

Roughly: $u(x) \sim \sum_{(z,k)\in E} a_{z,k} \cdot x^z \log^k x$ as $x \to 0$ $(x > 0)$

with derivatives.

Note: $x^z \log^k x = o\left(x^{z'} \log^{k'} x\right)$ as $x \to 0$

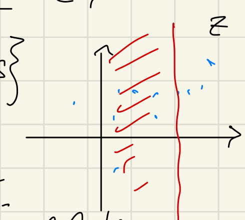
$\iff$ $\operatorname{Re} z > \operatorname{Re} z'$

or $\operatorname{Re} z = \operatorname{Re} z'$ and $k < k'$.

Def: $E \subset \mathbb{C} \times \mathbb{N}_0$ is an index set $:\iff$

(i) $E_{\le s} := \{(z,k) \in E : \operatorname{Re} z \le s\}$ is finite $\forall s \in \mathbb{R}$

(ii) $(z,k) \in E,\ l \le k \Rightarrow (z,l) \in E$.



E is called smooth index set if, in addition,

(iii) $(z,k) \in E \Rightarrow (z+1,k) \in E$.

Space of remainders:

Def: $\mathcal{A}^s(\mathbb{R}_+) := \{ u \in C^\infty(\mathbb{R}_+^\circ) : (x\partial_x)^N u = O(x^s)\ \forall N \}$

Rem: · Why $x\partial_x$?

What should "$u(x) = O(x^s)$ holds with derivatives"

One (natural) answer: $u'(x) = O(x^{s-1})$ etc.

(cf for convergent power series) $\Updownarrow$

$x\partial_x u = O(x^s)$

· even if we only required low regularity in the interior, the condition $(x\partial_x)^N u$ bounded implies u smooth in $\mathbb{R}_+^\circ$. (Sobolev embedding)

Def: Let E be an index set. $u \in C^\infty(\mathbb{R}_+^\circ)$ is <u>E-smooth</u> (or <u>polyhomogeneous with index set E</u>) if

$$u(x) \sim \sum_{(z,k)\in E} a_{z,k}\, x^z \log^k x \qquad \text{for certain } a_{z,k} \in \mathbb{C}$$

in the sense that for all $s \in \mathbb{R}$

$$u(x) = \sum_{(z,k)\in E_{\le s}} a_{kz}\, x^z \log^k x + r_s(x)$$

where $r_s \in \mathcal{A}^s(\mathbb{R}_+)$.

$\mathcal{A}^E(\mathbb{R}_+) := \{E\text{-smooth fcts on } \mathbb{R}_+\}$

Rem: · Equivalent variations of the def.:

$\sum_{\dots \in E_{<s}} \dots \qquad r_s \in \mathcal{A}^{s-\varepsilon}\ \forall \varepsilon > 0.$

· Given any function $s'(s)$, $s'(s) \to \infty$ as $s \to \infty$,

$\sum_{\dots \in E_s} \dots, \quad r_s \in \mathcal{A}^{s'}$

ex: $s' = s - 100$

ex: · $E = \mathbb{N}_0 \times \{0\} \Rightarrow \mathcal{A}^E(\mathbb{R}_+) = C^\infty(\mathbb{R}_+)$

Proof: "$\Rightarrow$": Taylor's theorem: $u(x) = u_0 + u_1 x + \dots + u_N x^N + x^{N+1} v(x)$, v smooth.

"$\Leftarrow$": $s = 0$: $u = a_0 + O(1) \Rightarrow u$ bounded.

$s = 1$: $x\partial_x u = x\partial_x(a_0 + x a_1) + O(x) \Rightarrow u'$ bounded

etc. $\Rightarrow u^{(k)}$ bounded $\forall k \Rightarrow u \in C^\infty(\mathbb{R}_+)$.

ex:

- $u(x) = \sin\frac{1}{x}$ is not phg (for any E).

formally: $\sin\frac{1}{x} = \frac{1}{x} - \frac{1}{3!}\frac{1}{x^3} + \dots$

- $\frac{1}{1 - x\log x} = 1 + x\log x + x^2\log^2 x + \dots$

Lemma: $\mathcal{A}^s, \mathcal{A}^E$ are vector spaces closed under $x\partial_x$.

$$(x\partial_x)(x^z \log^k x) = z\, x^z \log^k x + k \cdot x^z \log^{k-1} x$$

$$(x\partial_x - z)\, x^z \log^k x = k \cdot x^z \log^{k-1} x$$