Geodesics on singular surfaces

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Let $X \subset \mathbb{R}^n$ be a real algebraic surface. Let $p \in X$ be an isolated singularity of X, and let X_{reg} be the regular part of X. Assume p is not an isolated point of X. By considering shortest curves from p to nearby points, it is easy to see that there are many geodesics on X_{reg} ending at p, see for example [BL04]. There it is also shown that any such geodesic has a limit direction at p. We study the higher regularity properties of these geodesics; moreover, we study the exponential map based at p, that is, the whole family of geodesics ending at p. Simple examples show that such a geodesic may have no second derivative at p. However, one may hope for a complete asymptotic expansion in which fractional powers and possibly logarithmic terms occur. We prove that for a certain class of singular surfaces such complete expansions exist, in a suitable sense, for the exponential map.

Applications of such a detailed description of the exponential map are, among others:

- A notion of normal coordinates based at the singularity, with all the uses that normal coordinates have, for example in the analysis of the geometric partial differential operators on X_{reg} and of the diffraction of waves by the singularity.
- Complete asymptotic expansions of the volume of small balls, $B_r(p) = \{q : \mathrm{dist}(p,q) < r\}$, for $r \to 0$. Here, dist is the intrinsic distance on X. Once the existence of such an expansion is established, an interesting question would be to relate the coefficients of such an expansion to generalized notions of curvature of X at p, as is well-known in the case of a smooth point p.

Apart from trivial cases, a precise description of the local Riemannian geometry near a singularity is, to the best of our knowledge, only known in the case of asymptotically conical singularities: For this case, the exponential map is analyzed completely in [MW04], and this is then used in the analysis of the propagation of singularities for the wave equation on manifolds with conical singularities.

In our approach we analyze directly the system of ordinary differential equations (ODEs) describing the geodesics. To do this, one needs to use suitable coordinate systems. In the case of an isolated singularity it is natural to use polar coordinates centered at the singularity. Geometrically, introducing polar coordinates corresponds to blowing up the space X at the point p. It is known that by repeated blow-ups, a smooth space X' can be obtained (resolution of singularities); however, the metric on X' corresponding to the metric on X is degenerate at the exponential divisor (the preimage of p), and hence the coefficients of the ODEs blow up there. The problem is to deal with these singularities.

Our guiding idea is that geodesic flow should behave rather regularly when considered on a suitably blown-up space and with a suitable rescaling of time.

The existence of such a blown-up space is by far not obvious. Roughly speaking, the problem is to find a path between two opposing forces:

- with each blow-up the metric becomes more degenerate, hence the geodesic equations become more singular
- one needs to make sufficiently many blow-ups to resolve the singularity as well as the metric (here, 'resolution of a metric' needs to be defined; essentially, this means that the degeneration of the metric tensor at the exceptional divisor has a monomial normal form; for surfaces, this may be taken to be the normal form derived by Hsiang and Pati [HP85])

We are able to carry out this program for the following class of surfaces, modelled on a $(1, 2, 1)$-quasihomogeneous algebraic singularity: Let $C \subset \mathbb{R}^2$ be a smooth simple closed curve and set $\tilde{X} = C \times \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$. Let

$$\beta : \mathbb{R}^3 \to \mathbb{R}^3, \quad (u, v, z) \mapsto (uz, vz^2, z), \qquad X := \beta(\tilde{X}) \subset \mathbb{R}^3. \tag{1}$$

That is, the singularity of X is at the origin and quasihomogeneous of type $(1, 2, 1)$. $\tilde{X}$ is the blow-up (resolution) of X by the quasihomogeneous blow-down map β. Note that X is not necessarily algebraic. We can also allow higher order perturbations of X.

We make the following assumptions on C. Let

$$C_0 = \{q \in C : \ i^* du = 0 \text{ at } q\}, \quad \text{where } i : C \mapsto \mathbb{R}^2 \text{ is inclusion.}$$

We assume

$$(u, v) \in C_0 \Longrightarrow v = 0 \text{ and } C \text{ is nondegenerate at } (u, v). \tag{2}$$

Thus, the only points where the tangent to C is parallel to the v-axis lie on the u-axis, and C has only first order contact with its tangent there.

An example is the surface

$$X = \{(x, y, z) \in \mathbb{R}^3 : \ \left(\frac{x}{z}\right) + \left(\frac{y}{z^2}\right) = 1, \ z > 0\} \cup \{(0, 0, 0)\}$$

for which $C = \{u^2 + v^2 = 1\}$, the circle. Here $C_0 = \{(\pm 1, 0)\}$.

We consider the metric on X induced by the Euclidean metric on $\mathbb{R}^3$. This induces a smooth semi-Riemannian metric g on $\tilde{X}$ which is Riemannian in the interior of $\tilde{X}$.

Our first theorem shows the existence of the exponential map based at the singular point 0. This will be expressed in terms of geodesics on $\tilde{X}$ rather than X. We call a curve $\gamma : [0, T) \to \tilde{X}$ with $\gamma(0) \in \partial \tilde{X}$ and $\gamma(t)$ in the interior of $\tilde{X}$ for $t > 0$ a *geodesic* if $\gamma_{|(0,T)}$ is a geodesic with respect to the Riemannian metric g and γ is continuous at $t = 0$. We say γ starts at $\gamma(0)$.

Theorem 1. *Let C and $\tilde{X}$ be given as above.*

(a) *For each $q \in C$ there is a unique geodesic $\gamma_q : \mathbb{R}_+ \to \tilde{X}$ starting at $(q, 0)$.*
(b) *The map $\exp : C \times \mathbb{R}_+ \to \tilde{X}$, $(q, t) \mapsto \gamma_q(t)$ is a homeomorphism.*

Our main result is a precise description of the analytic properties of exp. To state this, it is more convenient to consider the inverse map

$$\exp^{-1} = (I, d) : \tilde{X} \to C \times \mathbb{R}_+.$$

Here, if $p \in \tilde{X}$ then $I(p)$ is the unique point $q \in C$ for which there is a geodesic from q to p, and $d(p)$ is the time it takes to get from q to p. In terms of X, $I(p)$ corresponds to the direction in which the unique geodesic from $\beta(p)$ to the singular point 0 arrives at 0, and $d(p)$ is the distance from $\beta(p)$ to 0.

Let

$$\pi : X' = [\tilde{X}, C_0 \times \{0\}, C_0 \times \mathbb{R}_+] \to \tilde{X}$$

be the iterated blow-up of $\tilde{X}$ in the (isolated) points of $C_0 \times \{0\}$, followed by a blow-up in the (preimage of) the lines $C_0 \times \mathbb{R}_+$. While the blow-up $\beta : \tilde{X} \to X$ resolves X in the sense of manifolds, the blow-up $[\tilde{X}, C_0 \times \{0\}] \to \tilde{X}$ resolves the metric (for example, in the sense of Hsiang-Pati). The last blow-up, of the lines $C_0 \times \mathbb{R}_+$, is needed for the analytic description of the exponential map below: It replaces each of these lines by two copies of itself (making them boundary lines) and has only the effect that non-smooth behavior is permitted transversal to these lines.

We denote the faces of X' as follows: C_{conic} is the preimage of $(C \setminus C_0) \times \{0\}$, ff is the front face of the first blow-up, that is, the preimage of $C_0 \times \{0\}$, and Z is the preimage of $C_0 \times (0, \infty)$.

Theorem 2. *π^*I and π^*d are polyhomogeneous conormal functions on X'. They are smooth at C_{conic} and* ff *but their expansion at Z contains half integral powers and may contain logarithms.*

Whether or not the logarithmic terms actually appear remains to be checked, but at present seems very likely. The question of their presence (for the case of semi-algebraic X) is interesting since an affirmative answer would imply that *Hardt's conjecture*, stating that the distance function on a semialgebraic set be subanalytic, is false. In fact, Theorem 2 suggests a *replacement for Hardt's conjecture*, namely that the distance function on a semi-algebraic set is conormal on a suitably blown-up space.

Outline of the proof:

We analyze directly the Hamiltonian flow describing the geodesics in the smooth part of X, that is, the interior of $\tilde{X}$, uniformly up to $\partial\tilde{X}$. On $\partial\tilde{X}$, the Hamilton vector field W is singular (i.e. blows up), and even more so at the points in $C_0 \times \{0\} \subset \partial\tilde{X}$. This singularity is resolved in three ways:

- rescaling the cotangent bundle of $\tilde{X}$, to the conic cotangent bundle ${}^cT^*\tilde{X}$
- rescaling time, by considering $V = zW$ instead of the Hamilton field W, where z is as in (1).
- blowing up certain points in ${}^cS^*\tilde{X}$ (the cosphere bundle corresponding to ${}^cT^*\tilde{X}$), lying over $C_0 \times \{0\}$, with suitable quasi-homogeneity.

The first two steps were already used by Melrose and Wunsch in the conic case, where they showed that (the pull-back of) V is a smooth vector field on ${}^cS^*\tilde{X}$ with non-degenerate hyperbolic critical points along a curve L which projects diffeomorphically to $\partial\tilde{X}$, under the natural projection $\pi_{\tilde{X}} : {}^cS^*\tilde{X} \to \tilde{X}$. The invariant manifold theorem can then be used to deduce the existence and smoothness of the exponential map in the conic case.

In our quasihomogenous situation, the metric on $\tilde{X}$ is conical except at $C_0 \times \{0\}$, and this yields additional singularities of V there. Therefore, an additional blow-up is needed. The main (and highly non-obvious) point of the proof is that these singularities can be resolved by blowing up the points in $(\pi_{\tilde{X}|L})^{-1}(C_0 \times \{0\})$ with a $1, 1, 3$-quasihomogeneity, and that the pull-back of V under that blow-up is not too degenerate. More precisely, one obtains a smooth vector field tangent to the boundary which has only hyperbolic critical points (at least in the regions that matter for the exponential map). The flow near these critical points, and hence everywhere, can then be analyzed.

The emergence of logarithms is related to the fact that the hyperbolic critical points are resonant and therefore (most likely) do not have smooth linearizations (but rather 'log-smooth' linearizations).

While our result is restricted to a special class of singularities, it is the first detailed (that is, to higher order) investigation of the inner geometry of algebraic sets beyond the conic case. Since for general algebraic surfaces a normal form of the metric is known (see [HP85]), we conjecture that similar results will be true in this more general context.

References

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