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QUASIISOMETRY OF SINGULAR METRICS

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Abstract. We investigate when two Riemannian metrics, defined near zero in $\mathbb{R}^n$ and possibly singular at zero, are quasiisometric via a coordinate change that may be singular at zero.

A *singular metric* is a Riemannian metric on $U \setminus 0$, for some neighborhood U of 0 in $\mathbb{R}^n$. A *singular coordinate change* is a diffeomorphism $\phi : U \setminus 0 \to U' \setminus 0$, with U, U' neighborhoods of 0 in $\mathbb{R}^n$, which extends to a homeomorphism $U \to U'$ when one sets $\phi(0) = 0$. Two singular metrics g, g' are called *(strongly) quasiisometric* if there is a constant C such that

$$C^{-1} g_x \leq g'_x \leq C g_x$$

(as quadratic forms) for all x in some pointed neighborhood of zero; we then write $g \approxeq_C g'$ or simply $g \approxeq g'$. We call g, g' *weakly quasiisometric* (wqi in short) if there is a singular coordinate change ϕ such that

$$g \approxeq \phi^* g'.$$

In this case we write $g \approx_\phi g'$ or simply $g \approx g'$. Clearly, both $\approxeq$ and $\approx$ are equivalence relations.

In this note we initiate a study of the question when two singular metrics are wqi. In particular, we give conditions when a metric on $\mathbb{R}^2 \setminus 0$ is wqi to a metric of the form $dr^2 + a(r)^2 d\theta^2$ (Propositions 2.3, 2.4), and when two such metrics are wqi to each other (Theorem 2.5). As a corollary, we show that two 'horns' $V_{\gamma_1}, V_{\gamma_2}$, where

$$V_\gamma = \{(x, y, z) : 0 \leq z < 1, \, z^\gamma = \sqrt{x^2 + y^2}\}$$

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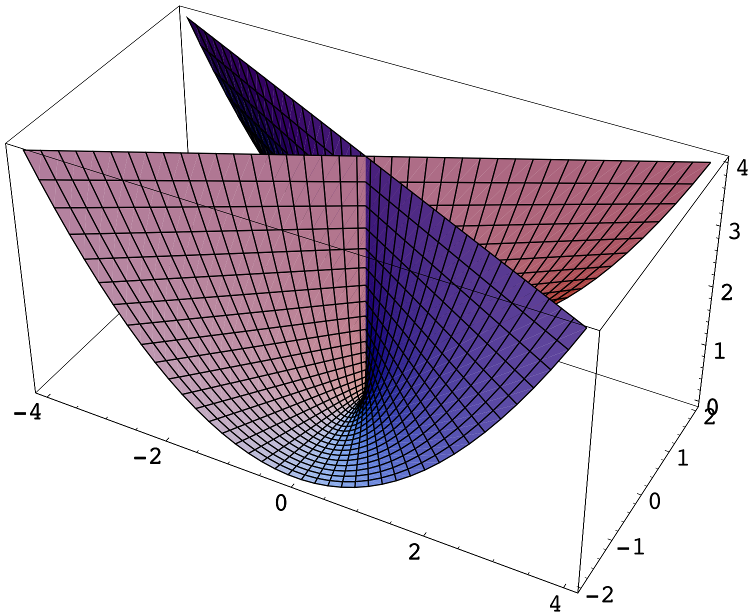

FIGURE 1. The real Whitney umbrella

for $\gamma \geq 1$, are wqi if and only if $\gamma_1 = \gamma_2$. Here we use the intrinsic metric on V_γ induced by the Euclidean Riemannian metric on $\mathbb{R}^3$, and the extension of the notion of wqi to these spaces is the obvious one. See Corollary 2.7.

Also, the (real) Whitney umbrella (without handle)

$$W = \{(x, y, z) : x^2 = y^2 z,\ z \geq 0\} \subset \mathbb{R}^3$$

(see Figure 1) with the induced metric is analyzed in detail, and we show that a neighborhood of zero in W is wqi to a cone (i.e. to V_1) when one considers the normalization of W, i.e. the space obtained by removing the intersection of the two sheets along $\{(0, 0, z) : z > 0\}$. More precisely, the normalization is the map

$$\mathbb{R}^2 \to W, \quad (u, v) \mapsto (uv, v, u^2) \in W.$$

This map is bijective except over the positive z-axis (which corresponds to $u \neq 0, v = 0$), where it is two-to-one. Pull-back of the Euclidean metric $dx^2 + dy^2 + dz^2$ under this map yields the smooth semi-Riemannian metric

$$g_W = (4u^2 + v^2)du^2 + 2uv\, dudv + (1 + u^2)dv^2 \tag{1}$$

on $\mathbb{R}^2$, which is Riemannian on $\mathbb{R}^2 \setminus 0$, i.e. a singular metric in the sense above. In Section 1 we show that this metric is wqi to the Euclidean metric on $\mathbb{R}^2$ (which is wqi to a cone, of course).

A problem that is closely related to ours is the local classification of semialgebraic or subanalytic subsets of $\mathbb{R}^n$ up to (weak) quasiisometry; here, one uses the intrinsic metric on such sets, defined by lengths of shortest curves inside the set. See [Bi], [G]. One result of these papers is that a neighborhood of an isolated singularity of any subanalytic surface is wqi to a finite union of spaces V_γ with various rational numbers $\gamma \geq 1$, which are glued at their tips. Such a classification is unknown in higher dimensions.

For two functions f, g on a set S we write

$$f \approxeq g$$

if there is a constant C such that for all $x \in S$, one has $C^{-1} f(x) \leq g(x) \leq C f(x)$.

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1. Metrics of Whitney umbrella type are weakly quasiisometrically Euclidean

In this section we will show that

$$g_W \approx g_{\mathrm{Eucl}} \tag{2}$$

holds for the Whitney umbrella metric g_W and the standard Euclidean metric g_{Eucl} on $\mathbb{R}^2$. The latter is clearly wqi to the infinite cylinder $\{x^2 + y^2 = z^2, z > 0\}$ in $\mathbb{R}^3$, via the projection onto the x, y-plane.

We need a little lemma, a generalization of which will be proved in the next section (see Lemma 2.1 and the remark following it):

Lemma 1.1. *Let $g = adx^2 + 2bdx\,dy + cdy^2$ be a Riemannian metric on a subset U of $\mathbb{R}^2$. Then g is (strongly) quasiisometric to the 'diagonal' metric $adx^2 + cdy^2$ if and only if there is $\varepsilon > 0$ such that*

$$b^2 \leq (1 - \varepsilon)ac \tag{3}$$

on U.

Now recall formula (1) for the Whitney umbrella metric. For $u \leq 1$ we have $u^2 \leq (1 + u^2)/2$, and this implies $(uv)^2 \leq \frac{1}{2}(4u^2 + v^2)(1 + u^2)$ for u, v near zero, so from Lemma 1.1 (with $\varepsilon = 1/2$) we see that

$$g_W \approxeq (4u^2 + v^2)du^2 + (1 + u^2)dv^2 \approxeq (u^2 + v^2)du^2 + dv^2.$$

In order to put this into the form $dx^2 + dy^2$, it seems natural to try the singular coordinate change $x = u\sqrt{u^2 + v^2}, y = v$. Then

$$\begin{aligned} dx &= (\sqrt{u^2 + v^2} + \frac{u^2}{\sqrt{u^2 + v^2}})du + \frac{uv}{\sqrt{u^2 + v^2}}dv \\ dy &= dv. \end{aligned}$$

By Lemma 1.1, a metric of the form

$$(a\,du + b\,dv)^2 + dv^2 = a^2\,du^2 + 2ab\,du\,dv + (1 + b^2)dv^2$$

is quasiisometric to its diagonal part if and only if $(ab)^2 \le (1 - \varepsilon)a^2(1 + b^2)$ for a constant $\varepsilon > 0$, which is equivalent to $|b| \le const$. In this case, one even gets

$$(a\,du + b\,dv)^2 + dv^2 \approxeq a^2\,du^2 + dv^2.$$

In the case at hand we have $|b| = \frac{|uv|}{\sqrt{u^2+v^2}} \le 1$ near zero. Also,

$$\frac{u^2}{\sqrt{u^2 + v^2}} \le \sqrt{u^2 + v^2}.$$

Therefore, we finally get

$$dx^2 + dy^2 \approxeq \left(\sqrt{u^2 + v^2} + \frac{u^2}{\sqrt{u^2 + v^2}}\right)^2 du^2 + dv^2 \approxeq (u^2 + v^2)\,du^2 + dv^2.$$

This proves (2). □

2. Partial classification of metrics with respect to weak quasiisometry

In this section, we will provide some criteria which allow to decide whether two given singular metrics are wqi.

We will always work in polar coordinates near zero in $\mathbb{R}^n$. Thus, we consider our metrics as defined on a cylinder

$$Z = Z_{r_0} = (0, r_0) \times S^{n-1}$$

for some $r_0 > 0$. A singular coordinate change is then a diffeomorphism ϕ between open subsets of Z_{r_0} containing strips of the form Z_{r_1} for some $r_1 < r_0$, satisfying the condition that

$$r \to 0 \text{ when } \rho \to 0, \text{ with } \phi : (\rho, \omega) \mapsto (r, \theta).$$

2.1. **Splitting off dr^2: When is a singular metric quasiisometric to a metric of the form $dr^2 + g^{S^{n-1}}(r)$?** The following lemma shows when a metric on a product is quasiisometric to its 'diagonal part':

Lemma 2.1. *Let M, N be manifolds and g a Riemannian metric on $M \times N$. Denote by $g^M + g^N$ the metric on $M \times N$ which at the point (m,n) equals g on T_mM and T_nN and for which these two subspaces of $T_{(m,n)}(M \times N)$ are orthogonal.*

Then

$$g \approxeq g^M + g^N$$

if and only if there is a constant $\delta > 0$ such that the angle

$$\angle_g(T_mM, T_nN) > \delta$$

for all $m \in M, n \in N$.

Remark: By definition, the angle condition means that for all m, n and $v \in T_mM, w \in T_nN$ one has

$$\arccos \frac{g(v,w)}{|v| \cdot |w|} > \delta$$

where $|v| := |v|_g := \sqrt{g(v,v)}$ etc., i.e.

$$|g(v,w)| < (1-\varepsilon)|v| \cdot |w| \tag{4}$$

with $1 - \varepsilon = \cos\delta$, which is (up to a square root) just the condition in Lemma 1.1.

Proof. Assume that (4) holds. We then have

$$\begin{aligned} |v+w|_g^2 &= |v|^2 + |w|^2 + 2g(v,w) \\ &\geq |v|^2 + |w|^2 - 2(1-\varepsilon)|v| \cdot |w| \\ &\geq \varepsilon(|v|^2 + |w|^2) \\ &= \varepsilon|v+w|^2_{g^M+g^N} \end{aligned}$$

and, using the parallelogram identity,

$$\begin{aligned} |v+w|_g^2 &\leq |v+w|_g^2 + |v-w|_g^2 \\ &= 2(|v|^2 + |w|^2) \\ &= 2|v+w|^2_{g^M+g^N}. \end{aligned}$$

This proves that $g \approxeq g^M + g^N$ with quasiisometry constant $\max(2, (1-\cos\delta)^{-1}) \sim 2\delta^{-2}$ for small δ.

The prove the converse, assume that $|g(v,w)| > (1-\varepsilon)|v|\cdot|w|$ for some ε and $v \in T_mM, w \in T_nN$. By multiplying v with a real number, we can assume $|v| = |w|$ and

$$g(v,w) < -(1-\varepsilon)|v|\cdot|w|.$$

Then one has

$$\begin{aligned} |v+w|_g^2 &< |v|^2 + |w|^2 - 2(1-\varepsilon)|v|\cdot|w| = 2\varepsilon|v|^2 \\ &= \varepsilon|v+w|^2_{g^M+g^N}. \end{aligned}$$

Therefore g and $g^M + g^N$ cannot be ε^{-1}-quasiisometric then. □

In the sequel we will write $|\ |_g$ for both the metric on the tangent bundle and the dual metric on the cotangent bundle.

For the special case that M is one-dimensional we now formulate several criteria that characterize when we are in the situation of the previous lemma and in addition g^M is of the simple form dr^2.

Lemma 2.2. *Let N be a manifold and g a metric on $X = (0,r_0)\times N$. Denote by r the coordinate on $(0,r_0)$. Then the following are equivalent:*

(i) *For a smooth family $g^N(r)$ of metrics on N, one has*

$$g \approxeq dr^2 + g^N(r),$$

(ii) *there is a constant C such that for all $x \in X$*

$$|dr_{|x}|_g < C \quad \textit{and} \quad |\partial_{r|x}|_g < C,$$

(iii) $|dr|_g \approxeq 1$ *and* $|\partial_r|_g \approxeq 1$ *on X,*

(iv) $|dr|_g \approxeq 1$ *and there is a constant $\varepsilon > 0$ such that for all $x = (r,y) \in X$*

$$\angle_g(\partial_{r|x}, T_yN) > \varepsilon.$$

Here, $\partial_{r|(r_0,y_0)} := \frac{d}{dt}_{|t=0}(r_0+t,y_0)$ is the first coordinate vector field.

Remarks:

1. With respect to the metric $adr^2 + 2bdrdy + cdy^2$ one has

$$|\partial_r|^2 = a \text{ and } |dr|^2 = c/(ac-b^2),$$

so the two conditions in (ii) are independent.

2. The form dr is invariant under a change of coordinates of the form $(r,y)\mapsto (r,\psi(r,y))$, i.e., it just depends on the choice of the function r on X, while the vector ∂_r is not.

PROOF. (i) $\Longrightarrow$ (ii) is clear since with respect to the metric $g' = dr^2 + g^N(r)$, one has

$$|dr|_{g'} \equiv 1, \quad |\partial_r|_{g'} \equiv 1.$$

(ii) $\Longrightarrow$ (iii) follows from $dr(\partial_r) = 1$, which implies $|dr|_g |\partial_r|_g \geq 1$.
For the implication (iii) $\Longrightarrow$ (iv), we have to show that $\angle_g(\partial_r, w) > \varepsilon$ for all $w \in TN$. If ∇r denotes the vector dual to the one form dr, then $\angle_g(\nabla r, w) = \pi/2$ for $w \in TN$ since $dr(w) = 0$. Therefore, we have to show $\angle_g(\partial_r, \nabla r) < \pi/2 - \varepsilon$. Now this follows from

$$\frac{g(\partial_r, \nabla r)}{|\partial_r||\nabla r|} = \frac{dr(\partial_r)}{|\partial_r||dr|} \geq C^{-2}.$$

Finally, we prove that (iv) implies (i): First, Lemma 2.1 shows that the second condition in (iv) implies $g \approxeq g^{(0,r_0)} + g^N$. But then $g^{(0,r_0)} \approxeq dr^2$ just means $|dr|_g \approxeq 1$, so the proof is complete. □

Applying Lemma 2.2 to the case of $\mathbb{R}^n$ we obtain an answer to the question in the title of this subsection. We only reformulate one of the characterizations of the lemma.

Proposition 2.3. *Let U be a pointed neighborhood of the origin in $\mathbb{R}^n$ and g a metric on U. Let*

$$(r, \theta) : U \to Z_{r_0}$$

be some coordinatization of U with $r(x) \to 0$ for $x \to 0$, and $\tilde{g}$ the induced metric on Z_{r_0}.

Then

$$\tilde{g} \approxeq dr^2 + g^{S^{n-1}}(r), \tag{5}$$

for some smooth (in $r > 0$) family of metrics on S^{n-1}, if and only if $|dr|_g \approxeq 1$ and the angle between the hypersurfaces $r = \text{const}$ and the curves $\theta = \text{const}$ is bounded away from zero.

Note that in the nonsingular case, i.e. if g extends to a smooth metric on $U \cup \{0\}$ then one can take normal polar coordinates (r, θ) with respect to zero, in particular $r(x) = \text{dist}_g(0, x)$. Then $|dr| = 1$, and the well known Gauss Lemma says that the spheres $\{r = \text{const}\}$ are perpendicular to the rays $\theta = \text{const}$, and this means that the metric has exactly the form (5). Also, in this case the family $g^{S^{n-1}}(r)$ extends smoothly to $r = 0$.

2.2. **Existence of a 'good' parametrization of S^{n-1}, for $n = 2$.** Having treated the r-part of parametrizations satisfying (5) in some detail, we now look at the map θ parametrizing the level sets of r. Assuming the existence of some θ-parametrization for which (5) is satisfied, we ask how we can change it so that the family of metrics $g^{S^{n-1}}(r)$ looks 'simple'.

We now restrict our considerations to the two-dimensional case, i.e. $N = S^1 = \mathbb{R}/2\pi\mathbb{Z}$. A metric on S^1, depending on the parameter r, is of the form

$$g^{S^1}(r) = H(r,\theta)^2 d\theta^2 \tag{6}$$

with some smooth positive function H on Z. We want to see if H can be made to depend on r only, via a singular coordinate change. If H depends on r only then $2\pi H(r)$ is the length of the curve $\{r\} \times S^1$, and θ measures (normalized) arc length (from some reference point $\theta = 0$). This suggests, in the general case, reparametrizing S^1 by arclength $\tilde{\tau}$ or normalized arclength τ for each fixed r, i.e. considering

$$\tilde{\tau}(r,\theta) = \int_0^\theta H(r,\omega)\, d\omega \tag{7}$$

$$l(r) = \tilde{\tau}(r, 2\pi) \quad \text{and} \tag{8}$$

$$\tau(r,\theta) = 2\pi \frac{\tilde{\tau}(r,\theta)}{l(r)}. \tag{9}$$

Then the metric on $\{r\} \times S^1$ has the simpler form

$$g^{S^1}(r) = \left(\frac{l(r)}{2\pi}\right)^2 d\tau^2.$$

However, this singular coordinate change $(r,\theta) \to (r,\tau)$ introduces mixed terms in the full metric g. The following proposition states the conditions when these mixed terms can be neglected.

Proposition 2.4. *On $Z = (0, r_0) \times S^1$ consider the metric*

$$g = dr^2 + H(r,\theta)^2 d\theta^2.$$

Let a be a positive function on $(0, r_0)$ and $t = t(r,\theta)$ an r-dependent reparametrization of S^1. Then

$$g \approxeq dr^2 + a(r)^2 dt^2 \tag{10}$$

if and only if there is a constant C such that on Z one has

$$a|t_r| < C \tag{11}$$

$$a t_\theta \approxeq H. \tag{12}$$

In particular, it then follows that

$$a \approxeq l.$$

For the (normalized) arclength reparametrization this means that

$$g \approxeq dr^2 + l(r)^2 d\tau^2$$

if and only if

$$l|\tau_r| < C. \tag{13}$$

Finally, this condition is a consequence of the following condition:

$$\int_0^{2\pi} |\frac{\partial H}{\partial r}(r,\theta)|\, d\theta < C.$$

Remark It is possible that arclength parametrization $t = \tau$ does not work but some other parametrization t does. For example, one could have $H \approxeq 1$ but with a 'bad' r-dependence (as $r \to 0$) for certain θ, e.g. like $1 + \sqrt{r}$ on some θ-interval, and with $l(r) \equiv 2\pi$. Then (13) is violated but clearly $g \approxeq dr^2 + d\theta^2$, so $t = \theta$ would work.

Proof. We have

$$dr^2 + a^2 dt^2 = dr^2 + a^2(t_r dr + t_\theta d\theta)^2.$$

Arguing in the same way as in the second application of Lemma 1.1 in the proof of (2), one sees that this is quasiisometric to a diagonal metric if and only if (11) is satisfied, and that this diagonal metric can then be taken as $dr^2 + a^2 t_\theta^2 d\theta^2$. From this, the first claim follows.

Integrating $at_\theta \approxeq H$ over S^1 gives $a \approxeq l$. If $t = \tau$ then one has $l\tau_\theta = 2\pi H$ from (7), (9), so the second condition (12) is satisfied. The final claim follows from

$$|l\tau_r/2\pi| = |\tilde{\tau}_r - \frac{l'}{l}\tilde{\tau}| \le |\tilde{\tau}_r| + |l'| \le 2\int_0^{2\pi} |H_r|\, d\theta.$$

□

2.3. Recovering the function a from the weak quasiisometry class of $dr^2 + a(r)^2 d\theta^2$. We now ask to what extent the factor a is determined by the weak quasiisometry class of the metric $dr^2 + a(r)^2 d\theta^2$. Clearly, it is determined up to a multiple bounded above and away from zero if we only allow the metric to change in its *strong* quasiisometry class. Also, (the easiest part of) Proposition 2.4 shows that it is determined just as much if we allow singular coordinate changes that preserve the r-coordinate. The following theorem shows that, in general, something slightly weaker is true.

Theorem 2.5. *Let $a, \bar{a}$ be two positive increasing functions on $(0, \varepsilon)$, and assume that*

$$dr^2 + a(r)^2 d\theta^2 \approx dr^2 + \bar{a}(r)^2 d\theta^2 \tag{14}$$

on $Z_\varepsilon = (0, \varepsilon) \times S^1$. Then

$$\frac{1}{C} a(\frac{r}{C}) \quad \leq \quad \bar{a}(r) \quad \leq \quad Ca(Cr) \tag{15}$$

for all r near zero, where C is the constant implicit in (14).

Remark Clearly, there is a converse statement if a and $\bar{a}$ satisfy a doubling condition, i.e. if there is a constant C' such that for all r we have $a(2r) \leq C'a(r)$, $\bar{a}(2r) \leq C'\bar{a}(r)$. In this case, (15) even implies strong quasiisometry in (14) (with a different C).

Proof. The assumption (14) means that there is a singular change of coordinates $\phi : (r, \theta) \mapsto (\bar{r}, \bar{\theta})$ such that

$$dr^2 + a(r)^2 d\theta^2 \approxeq_C d\bar{r}^2 + \bar{a}(\bar{r})^2 d\bar{\theta}^2. \tag{16}$$

The idea of the proof is that this quasiisometry implies that level curves of r and $\bar{r}$ at comparable levels should have comparable length. These lengths are given by $2\pi a$ and $2\pi\bar{a}$ respectively.

Denote $g = dr^2 + a^2 d\theta^2$ and $\bar{g} = d\bar{r}^2 + \bar{a}^2 d\bar{\theta}^2$. Fix $\rho > 0$ such that Z_ρ is contained in the domain and range of ϕ. By symmetry, it suffices to prove

$$a(\frac{\rho}{C}) \leq C\bar{a}(\rho). \tag{17}$$

Consider $\bar{r}, \bar{\theta}$ as functions of (r, θ), which are standard coordinates on Z_ε. First, we prove that

$$\bar{r} \leq Cr. \tag{18}$$

To prove this, note that $d\bar{r} = \bar{r}_r dr + \bar{r}_\theta d\theta$ and quasiisometry imply $|\bar{r}_r| \leq |d\bar{r}|_g \leq C|d\bar{r}|_{\bar{g}} = C$, and because $\bar{r} = 0$ for $r = 0$, this gives

$$\bar{r}(r, \theta) = \int_0^r \bar{r}_r(s, \theta)\, ds \leq Cr$$

proving (18).

(18) means that the level curve $\bar{r} = \rho$ lies completely 'to the right' of the vertical level curve $r = \rho/C$, see Figure 2. Denoting by $L_g(\gamma)$ the length of a

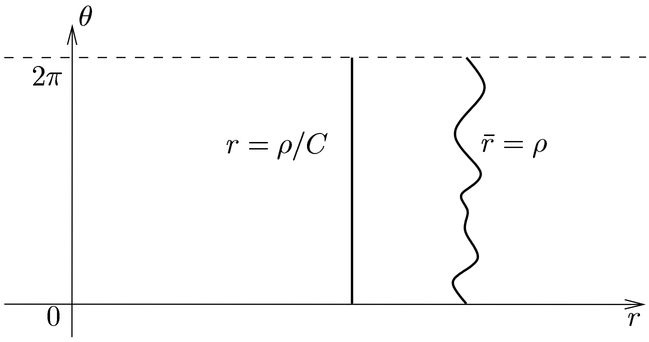

FIGURE 2. Level curves of r and $\bar{r}$

curve γ with respect to a metric g, we will prove below that monotonicity of a then implies

$$L_g(r = \rho/C) \leq L_g(\bar{r} = \rho). \tag{19}$$

From this we easily deduce (17):

$$2\pi a(\rho/C) = L_g(r = \rho/C) \leq L_g(\bar{r} = \rho) \leq CL_{\bar{g}}(\bar{r} = \rho) = 2\pi C\bar{a}(\rho),$$

where the second inequality comes from the quasiisometry (16).

It remains to prove (19). The path $\gamma : S^1 \to (0, \varepsilon) \times S^1, \bar{\theta} \mapsto \phi^{-1}(\rho, \bar{\theta})$ parametrizes the level curve $\bar{r} = \rho$. Write it in components as $\gamma = (r, \theta)$. Since ϕ is a diffeomorphism between two annuli, it has degree one or minus one. Since the degree of $\theta : S^1 \to S^1$ is continuous and therefore constant in ρ, it must also be one or minus one. This implies

$$\int_0^{2\pi} |\dot{\theta}(\bar{\theta})|\, d\bar{\theta} \geq 2\pi$$

where a dot means the derivative with respect to the curve parameter $\bar{\theta}$. Now the definition of g gives $|\dot{\gamma}|_g \geq a(r)|\dot{\theta}|$. From (18) we see that, along γ, $r \geq \rho/C$, so $a(r) \geq a(\rho/C)$ by monotonicity. Altogether, we deduce

$$L_g(\bar{r} = \rho) = \int_0^{2\pi} |\dot{\gamma}|_g d\bar{\theta} \geq a(\rho/C) \int_0^{2\pi} |\dot{\theta}| d\bar{\theta} \geq 2\pi a(\rho/C) = L_g(r = \rho/C).$$

This finishes the proof of (19) and of the theorem. □

Example 2.6. Suppose $\alpha, \beta > 0$ and

$$dr^2 + r^\alpha \theta^2 \approx dr^2 + r^\beta d\theta^2.$$

The proposition gives $C_0^{-1}r^\alpha \le r^\beta \le C_0 r^\alpha$ for some constant C_0 and all r near 0, and this implies $\alpha = \beta$.

Corollary 2.7. *The spaces $V_{\gamma_1}, V_{\gamma_2}$ ($\gamma_1, \gamma_2 > 0$) are wqi if and only if either $\gamma_1 = \gamma_2$ or both γ_1, γ_2 are ≤ 1.*

Proof. Denote by g_γ the metric on V_γ, i.e. $g_\gamma = (dx^2 + dy^2 + dz^2)_{|V_\gamma}$.

First, assume $\gamma \ge 1$. Using the parametrization

$$\phi : (r, \theta) \in \mathbb{R}_+ \times S^1 \mapsto (r^\gamma \cos\theta, r^\gamma \sin\theta, r) \in V_\gamma$$

we obtain for the metric

$$\phi^* g_\gamma = (1 + \gamma^2 r^{2(\gamma-1)})dr^2 + r^{2\gamma} d\theta^2.$$

Since $\gamma \ge 1$, this is clearly (strongly) quasiisometric to $dr^2 + r^{2\gamma}d\theta^2$ in $\{r < 1\}$, and by Example 2.6 the weak quasiisometry class of this metric determines γ uniquely.

For $\gamma < 1$ use the parametrization

$$\psi : (x, y) \in \mathbb{R}^2 \mapsto (x, y, f(x, y)) \in V_\gamma, \quad f(x, y) = (x^2 + y^2)^{1/2\gamma}.$$

Then

$$\psi^* g_\gamma = (1 + f_x^2)dx^2 + 2 f_x f_y \, dxdy + (1 + f_y^2)dy^2$$

which is quasiisometric to $dx^2 + dy^2$ iff f_x and f_y are bounded, by Lemma 1.1. Since ∇f is bounded near zero for $\gamma \le 1$, we see that all the V_γ with $\gamma \le 1$ are wqi. □

A different proof of this corollary was given by L. Birbrair ([Bi]).

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