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Note

Counting Complements in the Partition Lattice, and Hypertrees

DANIEL GRIESER*

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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Given two partitions π, σ of the set $[n] = \{1, ..., n\}$ we call π and σ complements if their only common refinement is the partition $\{\{1\}, ..., \{n\}\}$ and the only partition refined by both π and σ is $\{[n]\}$. If $\pi = \{A_1, ..., A_m\}$ then we write $|\pi| = m$. We prove that the number of complements σ of π satisfying $|\sigma| = n - m + 1$ is

$$\prod_{i=1}^{m} |A_i| \cdot (n-m+1)^{m-2}.$$

For the proof we assign to each σ a hypertree describing the pattern of intersections of blocks of π and σ and then count the number of hypertrees and the number of σ corresponding to each hypertree.

1. Problem and Sketch of Solution

A *partition* $\pi = \{A_1, ..., A_m\}$ of the set $[n] = \{1, ..., n\}$ is an (unordered) family of nonempty subsets $A_1, ..., A_m$ of $[n]$ which are pairwise disjoint and whose union is $[n]$. We call the A_i the *blocks* of π, and let $|\pi| = m$. A partition $\{B_1, ..., B_r\}$ is a *refinement* of $\{A_1, ..., A_m\}$ if each B_j lies in some A_i. It is well known (but of no relevance in this paper) that the ordering relation so defined on the set of all partitions of $[n]$ makes it into a lattice. Two partitions π and σ of $[n]$ are *complements* if their only common refinement is $\{\{1\}, ..., \{n\}\}$ (we then write $\pi \wedge \sigma = \hat{0}$) and the only partition refined by both π and σ is $\{[n]\}$ (we then write $\pi \vee \sigma = \hat{1}$). We will prove:

* Current address: Department of Mathematics, University of California at Los Angeles, Los Angeles, California 90024.

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THEOREM 1. *If $\pi = \{A_1, ..., A_m\}$ is a partition of $[n]$, then the number of complements σ of π with $|\sigma| = n - m + 1$ is*

$$\prod_{i=1}^{m} |A_i| \cdot (n-m+1)^{m-2}. \tag{1}$$

Throughout the paper, $\pi = \{A_1, ..., A_m\}$ will be a fixed partition of $[n]$, and we denote by Σ the set of partitions σ counted in the theorem. In order to prove (1) we will split up Σ into pieces and count the pieces and their cardinalities. We will assign to each $\sigma \in \Sigma$ a hypertree H_σ, and one piece will be the set of σ belonging to one fixed hypertree H. It will turn out that the cardinality of the piece depends only and in a simple manner on the vertex degrees (the degree sequence) of H. We will give a formula for the number of hypertrees with a fixed degree sequence, and summing over all degree sequences will yield (1).

The relevant definitions and facts about hypertrees are given in Section 2, and the proof of Theorem 1 in Section 3. Some remarks as to possible simplifications and generalizations conclude the paper.

2. HYPERTREES

A *hypergraph* $H = (V, E)$ consists of a finite *vertex* set V and a finite family E of nonempty subsets of V, the set of *edges*. E may contain multiple elements (*multiple edges*). A one-element edge is a singleton or *loop*. The *degree* of vertex v is the number of edges containing v. A *path* from vertex v to vertex w is a sequence $v = v_0, e_1, v_1, ..., e_r, v_r = w$ of distinct vertices $v_0, ..., v_r$ and distinct edges $e_1, ..., e_r$ such that each e_i contains its two neigbors. If $v = w$ but otherwise all vertices are distinct, and if $r \geqslant 2$, we speak of a *cycle*.

H is *connected* if there is a path between any two vertices, or, equivalently, if for any proper subset V' of V there is an edge intersecting both V' and $V - V'$. H is a *hypertree* if H is connected and contains neither loops nor cycles. In particular, a hypertree has no multiple edges.

The well-known fact that a connected graph on m vertices has at least $m - 1$ edges, and exactly $m - 1$ edges if and only if it is a tree, generalizes easily to:

LEMMA 1. *Let the hypergraph $H = (V, E)$ be connected and loop free. Then*

$$\sum_{e \in E} |e| \geqslant |V| + |E| - 1, \tag{2}$$

and equality holds if and only if H is a hypertree.

Proof. We replace each edge $e=\{v_1, ..., v_{|e|}\}$ by $|e|-1$ edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, ..., $\{v_{|e|-1}, v_{|e|}\}$ (numbering arbitrary). Obviously, the resulting graph is connected if and only if H is, and is a tree if and only if H is a hypertree. Now (2) is equivalent to

$$\sum_{e \in E} (|e|-1) \geqslant |V|-1,$$

and the assertion follows from the analogue for graphs. ∎

We remark that if in the hypergraph $H=(V, E)$ on the vertex set $V=[m]$ the vertices $1, ..., m$ have degrees $d_1, ..., d_m$, respectively, then $\sum_{e \in E} |e| = \sum_{i=1}^m d_i$.

The following formula appeared in [2], but for completeness we give a proof here.

LEMMA 2. *Let $m, k, d_1, ..., d_m$ be positive integers, $\sum_{i=1}^m d_i = m+k-1$. Then the number of hypertrees on the vertex set $[m]$ with k edges in which vertex i has degree d_i $(i=1, ..., m)$ is*

$$h(m, k; d_1, ..., d_m) = S_{m-1,k} \binom{k-1}{d_1-1 \cdots d_m-1},$$

where $S_{m-1,k}$ is the Stirling number of the second kind.

Proof. By induction on m. For $m=1$ or $k=1$ the claim is true. Let $m>1$, $k>1$.

If $k \geqslant m$ then $\sum_{e \in E} |e| = \sum_{i=1}^m d_i \leqslant 2k-1$, hence some edge must be a loop, and there is no hypertree. Hence assume $k \leqslant m-1$.

Then some d_i must be one. Let $d_1=1$, and let H be a hypertree with degrees $d_1, ..., d_m$, and let e be the edge containing vertex 1.

Assume first $|e|=2$, $e=\{1, i\}$. Let H' be the hypergraph obtained by removing vertex 1 and edge e from H. H' is a hypertree on $m-1$ vertices, with $k-1$ edges and vertex degrees $d_2, ..., d_i-1, ..., d_m$, and $d_i-1>0$ because $k>1$ and H' is connected. Hence there are $h(m-1, k-1; d_2, ..., d_i-1, ..., d_m)$ possible H''s, and because we can recover H from i and H' the number of H's with $|e|=2$ is

$$\sum_{i=2}^m h(m-1, k-1; d_2, ..., d_i-1, ..., d_m) = S_{m-2,k-1} \binom{k-1}{d_2-1 \cdots d_m-1}.$$

If $|e| > 2$ then we remove only vertex 1 and obtain a hypertree H' on $m-1$ vertices, with k edges and vertex degrees $d_2, ..., d_m$. We can recover H from the knowledge of H' and of the edge which contained vertex 1. Hence the number of H's with $|e| > 2$ is

$$kh(m-1, k; d_2, ..., d_m) = kS_{m-2,k} \binom{k-1}{d_2 - 1 \cdots d_m - 1}.$$

Now the recursion $S_{m-1,k} = S_{m-2,k-1} + kS_{m-2,k}$ yields the result. ∎

Although we will not need it here, we note the immediate

COROLLARY 1. *The number of hypertrees with m vertices and k edges is*

$$S_{m-1,k} m^{k-1}.$$

A hypertree is essentially the same as a graph all of whose blocks are complete graphs. Viewed in this way, the corollary follows also from the well-known block-tree-theorem which gives a formula for the number of graphs with prescribed blocks (see [1]).

3. SOLUTION

Let $\pi = \{A_1, ..., A_m\}$ be fixed and $\sigma = \{B_1, ..., B_r\}$ be any partition of $[n]$, with

$$|B_1| \geqslant |B_2| \geqslant \cdots \geqslant |B_k| > 1 = |B_{k+1}| = \cdots = |B_r|.$$

Let H_σ be the hypergraph with vertex set $[m]$ and edges $C_1, ..., C_k$, where

$$C_j = \{i \in [m] \mid A_i \cap B_j \neq \emptyset\} \qquad (j = 1, ..., k).$$

Thus we think of the blocks of π as vertices, and a set of vertices is an edge if for some $j \in [k]$, B_j has elements in common with exactly these vertices. Hence the number of edges of H_σ is just the number of nonsingleton blocks of σ.

LEMMA 3. *H_σ has the following properties:*

(i) *$|C_j| \leqslant |B_j|$ for $j = 1, ..., k$, and equality holds for all j iff $\pi \wedge \sigma = \hat{0}$.*

(ii) *H_σ is connected iff $\pi \vee \sigma = \hat{1}$.*

(iii) *If* $\pi \vee \sigma = \hat{1}$, *then* $|\sigma| \leqslant n+1-|\pi|$, *and equality holds iff* $\pi \wedge \sigma = \hat{0}$ *and* H_σ *is a hypertree.*

Proof. (i) $|C_j| \leqslant |B_j|$ is clear, and ($\pi \wedge \sigma = \hat{0} \Leftrightarrow |A_i \cap B_j| \leqslant 1$, $\forall i \in [m], j \in [k]$) implies the second part.

(ii) $\pi \vee \sigma = \hat{1} \Leftrightarrow$ there are no proper subsets $I \subset [m]$ and $J \subset [r]$ for which $\bigcup_{i \in I} A_i = \bigcup_{j \in J} B_j \Leftrightarrow$ for all proper subsets $I \subset [m]$ there is a C_j intersecting both I and $[m]-I \Leftrightarrow H_\sigma$ is connected.

(iii) If $\pi \vee \sigma = \hat{1}$ then H_σ is connected, hence $\sum_{j=1}^k |C_j| \geqslant m+k-1$ by Lemma 1. By (i), $\sum_{j=1}^k |C_j| \leqslant \sum_{j=1}^k |B_j| = n-(r-k)$, and we obtain $r \leqslant n+1-m$, with equality iff $\sum_{j=1}^k |C_j| = \sum_{j=1}^k |B_j|$ (hence $|C_j| = |B_j|$ and H_σ has no loops) and $\sum_{j=1}^k |C_j| = m+k-1$ which by Lemma 1 and (i) is equivalent to $\pi \wedge \sigma = \hat{0}$ and H_σ hypertree. ∎

Now we count the number of complements having the same hypertree.

LEMMA 4. *Given a hypertree* H, *there are* $\prod_{i=1}^m (a_i)_{d_i}$ *complements* σ *of* π *with* $H_\sigma = H$. *Here* d_i *is the degree of vertex* i *in* H, $a_i = |A_i|$ *and* $(\alpha)_\beta = \alpha(\alpha-1)\cdots(\alpha-\beta+1)$.

Proof. Let $C_1, ..., C_k$ be the edges of H. We obtain all σ by first picking an element from every A_i with $i \in C_1$, thus composing B_1, then picking elements for B_2, etc. In the end we will have picked d_i elements from A_i for every i, one after another. This is possible in $\prod_{i=1}^m (a_i)_{d_i}$ ways, and each way gives a different σ because H has no multiple edges. ∎

Proof of Theorem 1. We first count the number of complements σ, $|\sigma| = n-m+1$, with a fixed number k of nonsingleton blocks. By Lemma 3(iii) any complement σ for which H_σ is a hypertree has $n-m+1$ blocks, and by Lemmas 2 and 4 this number equals (with $p = \prod_{i=1}^m a_i$)

$$\begin{aligned}
&\sum_{\substack{d_1, ..., d_m \geqslant 1 \\ d_1 + \cdots + d_m = m+k-1}} S_{m-1,k} \binom{k-1}{d_1-1 \cdots d_m-1} \prod_{i=1}^m (a_i)_{d_i} \\
&= pS_{m-1,k} \sum_{\substack{e_1, ..., e_m \geqslant 0 \\ e_1 + \cdots + e_m = k-1}} \binom{k-1}{e_1 \cdots e_m} \prod_{i=1}^m (a_i - 1)_{e_i} \\
&= pS_{m-1,k} \left(\sum_{i=1}^m (a_i - 1) \right)_{k-1} \\
&= pS_{m-1,k} (n-m)_{k-1}.
\end{aligned} \tag{3}$$

Here we used

LEMMA 5. *For nonnegative integers* $r, m, u_1, ..., u_m$

$$(u_1 + \cdots + u_m)_r = \sum_{\substack{e_1, ..., e_m \geq 0 \\ e_1 + \cdots + e_m = r}} \binom{r}{e_1 \cdots e_m} \prod_{i=1}^{m} (u_i)_{e_i}.$$

Proof. Let U_i $(i = 1, ..., m)$ be sets, $|U_i| = u_i$, and $U = \bigcup_{i=1}^m U_i$. The left hand side counts the number of ways to pick r elements in order from U; the right hand side counts the same, first determining which of the r elements are to be picked from each U_i. ▮

Now summing over k yields

$$p \sum_{k=0}^{m-1} S_{m-1,k}(n-m)_{k-1}$$
$$= \frac{p}{n-m+1} \sum_{k=0}^{m-1} S_{m-1,k}(n-m+1)_k = p(n-m+1)^{m-2}$$

by a well-known formula. ▮

4. REMARKS

The proof rests on the formula for the number of hypertrees which was proved by induction. But it seems that a more direct combinatorial proof, avoiding induction, should be possible, perhaps a generalization of the Prüfer sequence procedure used to enumerate trees (see [3]). Also, formulae (3) and (1) appear to demand a more direct proof.

The more general problem to determine the number $C(\pi, r)$ of complements of π with r blocks seems to be much more difficult. By Lemma 3 we have $C(\pi, r) = 0$ if $r > n - m + 1$. If, for $r < n - m + 1$, we want to use the same method, we have to count hypergraphs containing cycles; also, some care would have to be taken because of the possible occurence of multiple edges (see proof of Lemma 4).

If we drop the condition $\pi \vee \sigma = \hat{1}$ and require only $\pi \wedge \sigma = \hat{0}$ and $|\sigma| = r$ the problem becomes much easier. Using inversion, we obtain for the number of these σ

$$S(\pi, r) = \frac{1}{r!} \sum_{l=0}^{r} \binom{r}{l} (-1)^{r-l} \prod_{A \in \pi} (l)_{|A|},$$

see [4], end of Section 3. It is also proved there that the polynomial $\sum_{r \geqslant 0} S(\pi, r)\, x^r$ has only real, nonpositive roots. It would be interesting to settle the corresponding problem for the numbers $C(\pi, r)$.

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