§ 6. The Schwartz-kernel theorem

Fundamental idea: Each $k \in C^\infty(\Omega_1 \times \Omega_2, \mathbb{C})$ defines

$$K : \mathcal{D}(\Omega_2) \longrightarrow \mathcal{D}'(\Omega_1)$$
$$\varphi \longmapsto [K\varphi : \psi \longmapsto K\varphi(\psi)]$$

by $K\varphi(\psi) := \langle k, \psi \otimes \varphi \rangle_{L^1(\Omega_1 \times \Omega_2)}$
$$= \int_{\Omega_1} (K\varphi)(x) \cdot \psi(x)\, dx$$

where $K\varphi(x) := \int_{\Omega_2} k(x,y)\, \varphi(y)\, dy$

$K\varphi \in C^\infty(\Omega_1)$ is identified with $T_{K\varphi} \in \mathcal{D}'(\Omega_1)$.

Claim: Any $K : \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ arises this way.

Theorem 6.1 Let $\Omega_1 \subseteq \mathbb{R}^p$, $\Omega_2 \subseteq \mathbb{R}^q$ open

(a) Any distribution $k \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ induces a continuous linear map $K : \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ via

$$\langle K\varphi, \psi \rangle := \langle k, \psi \otimes \varphi \rangle \qquad (*)$$

for any $\varphi \in \mathcal{D}(\Omega_2)$, $\psi \in \mathcal{D}(\Omega_1)$.

(b) Any cts. linear map $K : \mathcal{D}(\Omega_2) \to \mathcal{D}'(\Omega_1)$ admits a unique $k \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ st $(*)$ holds.

k is called the Schwartz-kernel of K

"The Schwartz-kernel thm"

Example from linear algebra:

- $(k(i,j))_{1\leq i,j\leq n}$ Matrix with $\mathbb{C}$-entries
- ν – counting measure on $\{1,\dots,n\}$
- Vektors = functions: $\{1,\dots,n\}\longrightarrow\mathbb{C}$

Then $Kf(i)=\sum_{j=1}^{n}k(i,j)f(j)=\int_{\{1,\dots,n\}}k(x,y)f(y)\,d\nu$

defines a linear mapping in $(\mathbb{C}^n)^*\cong\mathbb{C}^n$, ie $K:\mathbb{C}^n\to\mathbb{C}^n$.

Schwartz-kernel thm reduces to: Any linear $K:\mathbb{C}^n\longrightarrow\mathbb{C}^n$ may be represented by its "Schwartz-kernel" $(k_{ij})\subset M(n,\mathbb{C})$ a matrix!

Today we prove this result following some preparations.

Proposition 6.2 Let $s>p/2$. Then for $a\in\mathbb{R}^p$, $\delta_a\in H^{-s}(\mathbb{R}^p)$;

$$\mathbb{R}^p\ni a\longmapsto\delta_a\in H^{-s}(\mathbb{R}^p)$$

is continuous and for $f\in\mathcal{S}(\mathbb{R}^p)$ we have

$$f=\int_{\mathbb{R}^p}f(a)\,\delta_a\,da\quad \left[\langle f,\varphi\rangle=\int_{\mathbb{R}^p}f(a)\langle\delta_a,\varphi\rangle\,da\right]$$

as a "Bochner-integral" (generalization of Lebesgue-int. to functions with values in a Banach-space) in $H^{-s}(\mathbb{R}^p)$

Proof: $\langle\mathcal{F}\delta_a,\varphi\rangle=\langle\delta_a,\mathcal{F}\varphi\rangle$

$$=\int_{\mathbb{R}^p}e^{-i\langle a,\xi\rangle}\varphi(\xi)\,d\xi$$

$$\Rightarrow\mathcal{F}\delta_a=e^{-i\langle a,\cdot\rangle}$$

$$\Rightarrow \int_{\mathbb{R}^p} \underbrace{|\mathcal{F}\delta_a(\xi)|^2}_{=1} \cdot (1+|\xi|^2)^{-s}\, d\xi < \infty \text{ for } s > p/2.$$

This proves that indeed, $\delta_a \in H^{-s}(\mathbb{R}^p)$.

Continuity of $a \mapsto \delta_a$: Let $a_n \to a \in \mathbb{R}^p$. Then

$$\|\delta_{a_n} - \delta_a\|_{-s}^2 = \int_{\mathbb{R}^p} |e^{-i\langle a_n,\xi\rangle} - e^{-i\langle a,\xi\rangle}|^2 (1+|\xi|^2)^{-s}\, d\xi$$

- integrand converges pointwise (at each ξ) to 0 as $n\to\infty$.
- integrand is uniformly bded by $4\cdot(1+|\cdot|^2)^{-s} \in L^1(\mathbb{R}^p)$

$\Rightarrow$ by Lebesgue $\lim_{n\to\infty} \|\delta_{a_n} - \delta_a\|_{-s}^2 = 0$

Bochner-integral $\int_{\mathbb{R}^p} f(a)\delta_a\, da$:

⌈ A measurable function $f: \Omega \to B$ is Bochner-integrable if $\int_\Omega \|f\|_B\, d\mu < \infty$. Here $(\Omega, \mathcal{A}, \mu)$ is a complete measure space and $(B, \|\cdot\|)$ any Banach-space. ⌋

- $\mathbb{R}^p \ni a \mapsto f(a)\cdot\delta_a \in H^{-s}(\mathbb{R}^p)$ is continuous and hence measurable in $(\mathbb{R}^p, \mathcal{L}, d\lambda_p)$.
- $\int_{\mathbb{R}^p} \|f(a)\delta_a\|_{H^{-s}}\, da \le \underbrace{\|\delta_0\|_{H^{-s}}}_{=\|\delta_a\|_{H^{-s}} \text{ indep. of } a\in\mathbb{R}^p} \cdot \int_{\mathbb{R}^p} |f|\, d\lambda < \infty$

Hence $\int_{\mathbb{R}^p} f(a)\delta_a\, da$ exists as a Bochner-integral and for $\varphi \in \mathcal{S}(\mathbb{R}^p)$:

$$\left\langle \int_{\mathbb{R}^p} f(a)\,\delta_a\,da, \varphi \right\rangle = \int_{\mathbb{R}^p} f(a)\langle \delta_a, \varphi\rangle\,da$$

$$= \int_{\mathbb{R}^p} f(a)\varphi(a)\,da = \langle f, \varphi\rangle$$

$$\Rightarrow \int_{\mathbb{R}^p} f(a)\,\delta_a\,da = f \qquad \boxtimes$$

Remark: (Ü) $\boxed{\text{If } s > p/2 + k, \text{ then } a \mapsto \delta_a \in H^{-s} \text{ is } k\text{-times continuously differentiable.}}$

Proposition 6.3 Let $s > p/2 + l$, $t > q/2 + l$. Then for any

$$K : H^{-s}(\mathbb{R}^p) \longrightarrow H^t(\mathbb{R}^q)$$

a bded, linear operator; there exists a unique C^l-function

$$k \in C^l(\mathbb{R}^q \times \mathbb{R}^p)$$

such that for $f \in \mathcal{S}(\mathbb{R}^p)$:

$$Kf(x) = \int_{\mathbb{R}^p} k(x,y)\,f(y)\,dy$$

Proof: By assumption of bdedness of $K : H^{-s}(\mathbb{R}^p) \to H^t(\mathbb{R}^q)$:

$$H^{-t}(\mathbb{R}^q) \times H^{-s}(\mathbb{R}^p) \longrightarrow \mathbb{C}$$

$$(f, g) \longmapsto \langle Kg, f\rangle_{H^t, H^{-t}}$$

is bilinear and continuous.

By Prop. 6.2 and the remark after Prop 6.2:

$$\begin{aligned} \mathbb{R}^q \times \mathbb{R}^p &\longrightarrow H^{-t}(\mathbb{R}^q) \times H^{-s}(\mathbb{R}^p) \\ (x,y) &\longmapsto (\delta_x, \delta_y) \end{aligned}$$

is C^k (k-times continuously differentiable).
Hence the composition:

$$\begin{aligned} \mathbb{R}^q \times \mathbb{R}^p \longrightarrow H^{-t} \times H^{-s} &\longrightarrow \mathbb{C} \\ (x,y) &\longmapsto \langle K\delta_y, \delta_x \rangle_{H^t, H^{-t}} =: k(x,y) \end{aligned}$$

is a C^k-function.

Furthermore, for $f \in \mathcal{S}(\mathbb{R}^p)$, $g \in \mathcal{S}(\mathbb{R}^q)$:

$$\int_{\mathbb{R}^q} (Kf)(x)\, g(x)\, dx \underset{\substack{\uparrow \\ Kf \in H^t(\mathbb{R}^q)}}{=} \int_{\mathbb{R}^q} \langle Kf, \underset{\substack{\uparrow \\ \in H^{-t}(\mathbb{R}^q)}}{\delta_x} \rangle_{H^t, H^{-t}} g(x)\, dx$$

$$\underset{(\text{Prop } 6.2)}{=} \int_{\mathbb{R}^q} \int_{\mathbb{R}^p} \underbrace{\langle K\delta_y, \delta_x \rangle_{H^t, H^{-t}}}_{= k(x,y)} f(y)\, g(x)\, dy\, dx$$

$$= \int_{\mathbb{R}^q} \underbrace{\left(\int_{\mathbb{R}^p} k(x,y) f(y)\, dy \right)}_{\Rightarrow\, = (Kf)(x).} g(x)\, dx$$

$\boxed{}$

[Uniqueness follows from $\int \Delta k \cdot \varphi = 0$ for all $\varphi \in \mathcal{S}(\mathbb{R}^p) \Rightarrow \Delta k = 0$]

Proof of the Schwartz-kernel theorem

$X \subset \mathbb{R}^p$, $Y \subset \mathbb{R}^q$ open,

(1) $k \in \mathcal{D}'(Y \times X) \Rightarrow K : \mathcal{D}(X) \longrightarrow \mathcal{D}'(Y)$

$$\langle K\varphi, \psi \rangle := \langle k, \psi \otimes \varphi \rangle$$

is a cts linear map

(2) Each cts linear $K : \mathcal{D}(X) \to \mathcal{D}'(Y)$ defines unique $k \in \mathcal{D}'(Y \times X)$ st. $\langle K\varphi, \psi \rangle = \langle k, \psi \otimes \varphi \rangle$.

Proof: (1) By definition of the weak top. on $\mathcal{D}'(Y)$

$$\mathcal{D}(X) \ni \varphi \longmapsto K\varphi \in \mathcal{D}'(Y)$$

is cts if for each fixed $\psi \in \mathcal{D}(Y)$, $\varphi \mapsto (K\varphi)(\psi)$ ist cts. Hence we need to show for fixed $\psi \in \mathcal{D}(Y)$:

$$\mathcal{D}(X) \ni \varphi \longmapsto \langle k, \psi \otimes \varphi \rangle$$

is a distribution, ie cts.

Let $K \subset X$ be cpt; $L := \operatorname{supp} \psi$.
Since $k \in \mathcal{D}'(Y \times X)$, there exist $C > 0$, $N \in \mathbb{N}$ st

$$\forall \chi \in \mathcal{D}_{L \times K}(Y \times X) : |\langle k, \chi \rangle| \le C \cdot p_N(\chi)$$

$$\left[\text{recall } p_N(\chi) = \sum_{|\alpha| < N} \| D^\alpha \chi \|_{\infty, L \times K} \right]$$

In particular for $\varphi \in \mathcal{D}_K(X)$; $\psi \otimes \varphi \in \mathcal{D}_{L \times K}(Y \times X)$

$$\Rightarrow |\langle k, \psi \otimes \varphi \rangle| \le C \cdot p_N(\psi \otimes \varphi)$$
$$\le C \sum_{|\alpha| \le N} \| D^\alpha (\psi \otimes \varphi) \|_\infty$$

$$\leq c' \cdot \sum_{|\alpha|+|\beta|\leq N} \| D^\alpha \psi \|_\infty \cdot \| D^\beta \varphi \|_\infty$$

$$\leq c'' \cdot p_N(\psi) \cdot p_N(\varphi) \Rightarrow (k, \psi \otimes \varphi)$$

is continuous ~~for~~ on $\mathcal{D}_K(X)$ for each cpt $K \subset X$.

<u>This proves (1).</u>

(2). First we construct Schwartz-kernel for cts $K: \mathcal{E}(X) \to \mathcal{E}'(Y)$.

$$\beta: \mathcal{E}(X) \times \mathcal{E}(Y) \longrightarrow \mathbb{C}$$
$$(\varphi, \psi) \longmapsto \langle K\varphi, \psi \rangle$$

- for fixed ψ, $\beta(\cdot, \psi)$ is cts by def'n of weak top. of $\mathcal{E}'(Y)$ since this is exactly what cty of $K: \mathcal{E}(X) \to \mathcal{E}'(Y)$ means.
- for fixed φ, $\beta(\varphi, \cdot)$ is cts trivially

Hence β is separately cts $\Rightarrow$ cts by [Lesch-Skript, Prop A.3.2]

Hence for each $L \subset X$, $L' \subset Y$ cpt, there exist $C>0$, $N \in \mathbb{N}$ st

$$|\langle K\varphi, \psi \rangle| \leq C \cdot p_{N,L}(\varphi) \cdot p_{N,L'}(\psi)$$

(compare proof of (1))

where $p_{N,L}(\varphi) = \sum_{|\alpha|\leq N} \| D^\alpha \varphi \|_{\infty, L}$

$\Rightarrow$ β extends to a cts bilinear form on $C^N(X) \times C^N(Y)$

$\Rightarrow$ By Sobolev-embedding for $s > N + p/2$, $-t > N + q/2$

$$H^s(\mathbb{R}^p) \times H^{-t}(\mathbb{R}^q) \longrightarrow \mathbb{C}$$
$$(\varphi, \psi) \longmapsto \beta(\varphi|_X, \psi|_Y) \text{ is cts.}$$

In other words there exists a cts linear map

$$\tilde{K}: H^s(\mathbb{R}^p) \longrightarrow H^t(\mathbb{R}^q) \text{ st } \langle \tilde{K}\varphi, \psi \rangle_{H^t, H^{-t}} = \langle K\varphi|_X, \psi|_Y \rangle$$

" $(I+\Delta)^{-M}T$ " definiert als

$\mathcal{F}^{-1}\left[(1+\|\xi\|^2)^{-M} \mathcal{F}T \right]$ im Distributiven Sinn

Dann gilt für $\varphi \in \mathcal{D}(X)$:

$$\langle (I+\Delta)^{-M}T, (I+\Delta)^M\varphi \rangle$$

$$= \langle \mathcal{F}^{-1}[(1+\|\xi\|^2)^{-M}\mathcal{F}T], (I+\Delta)^M\varphi \rangle$$

$$= \langle [1+\|\xi\|^2]^{-M}\mathcal{F}T, \mathcal{F}^{-1}(I+\Delta)^M\varphi \rangle$$

$$= \langle \mathcal{F}T, [1+\|\xi\|^2]^{-M}\mathcal{F}^{-1}(I+\Delta)^M\varphi \rangle$$

$$= \langle T, \mathcal{F}\{[1+\|\xi\|^2]^{-M}\mathcal{F}^{-1}(I+\Delta)^M\varphi \rangle$$

$$= \langle T, \cancel{(I+\Delta)^{-M}}\cancel{(I+\Delta)^M}\varphi \rangle$$

$$= \langle T, \varphi \rangle.$$

We want to apply Prop 6.3 to

$$\tilde{K} : H^{-(-s)}(\mathbb{R}^p) \to H^{t}(\mathbb{R}^q)$$

$$(-s) < -N - p/2, \quad t < -N - q/2 .$$

However, since $(-s) \not> p/2 + k$, $t \not> q/2 + k$, there is no direct appl.

Observe:

$$(I+\Delta)^{-M} : H^{\gamma} \longrightarrow H^{\gamma + 2M}$$

$$\Rightarrow (I+\Delta)^{-M} \tilde{K} (I+\Delta)^{-M} : H^{s-2M} \longrightarrow H^{t+2M}$$

For $M \gg 0$ sufficiently large $(2M - s) > p/2$, $(t+2M) > q/2$ and hence by Prop 6.3

$(I+\Delta)^{-M} \tilde{K} (I+\Delta)^{-M}$ has a continuous

Schwartz-kernel $k_1 \in C(\mathbb{R}^q \times \mathbb{R}^p)$

For $\varphi \in \mathcal{D}(X)$, $\psi \in \mathcal{D}(Y)$ we find:

$$\langle K\varphi, \psi \rangle = \langle (I+\Delta)^{-M} \tilde{K} (I+\Delta)^{-M} \{ (I+\Delta)^{M} \varphi \}, (I+\Delta)^{M} \psi \rangle$$

$$= \langle k_1, \underbrace{(I+\Delta)^{M}}_{\Delta_x \text{ Laplace on } \mathbb{R}^p} \varphi \otimes \underbrace{(I+\Delta)^{M}}_{\Delta_y \text{ Laplace on } \mathbb{R}^q} \psi \rangle$$

$$= \langle (I+\Delta_y)^{M} (I+\Delta_x)^{M} k_1, \varphi \otimes \varphi \rangle$$

For $M \in \mathbb{N}$, $(I+\Delta)^{M}$ is a diff. op and

$$k := (I+\Delta_y)^{M} (I+\Delta_x)^{M} k_1 \in \mathcal{E}'(Y \times X)$$

is the Schwartz-kernel of K.

Now consider linear cts $K: \mathcal{D}(X) \to \mathcal{D}'(Y)$

$\Rightarrow$ $\forall \sigma \in \mathcal{D}(Y), \rho \in \mathcal{D}(X)$: $\sigma K \rho : \mathcal{E}(X) \to \mathcal{E}'(Y)$

$\Rightarrow$ By previous discussion $\sigma K \rho$ admits Schwartz-kernel

$$k_{\sigma,\rho} \in \mathcal{E}'(Y \times X)$$

Define $k \in \mathcal{D}'(Y \times X)$ for K as follows:

For $\chi \in \mathcal{D}(Y \times X)$ with $\operatorname{supp} \chi = L \subset Y \times X$ cpt, choose

$\sigma \in \mathcal{D}(Y), \rho \in \mathcal{D}(X)$ st

$$\sigma \otimes \rho \equiv 1 \text{ in a nbd of } L$$

(Check that this is possible)

Then define $\langle k, \chi \rangle := \langle k_{\sigma,\rho}, \chi \rangle$

Ü
- construction independent of σ, ρ-choices
- k Schwartz-kernel is unique [Hint: Use $\mathcal{D}(Y) \otimes \mathcal{D}(X) \subset \mathcal{D}(Y \times X)$ is dense]

Smoothing operators:

Proposition 6.4 (without proof)

(a) If $k \in \mathcal{E}(\Omega_2 \times \Omega_1)$, then $K: \mathcal{D}(\Omega_1) \to \mathcal{D}'(\Omega_2)$ extends continuously to $K: \mathcal{E}'(\Omega_1) \to \mathcal{E}(\Omega_2)$, ie an operator with smooth Schwartz-kernel smoothens out distr-ns.

with $(T \in \mathcal{E}'(\Omega_1))$

$$(KT)(x) := T(k(\cdot, x))$$

[if $T = T_f, f \in \mathcal{D}$, then $T(k(\cdot, x)) = \int k(x,y) f(y)\, dy$]

(b) The Schwartz-kernel of a cts linear $K: \mathcal{E}'(\Omega_1) \to \mathcal{E}(\Omega_2)$ is smooth.

$\hookrightarrow$ such operators with smooth k are called Smoothing!