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Generating unfavourable VaR scenarios with patchwork copulas

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Abstract: The central idea of the paper is to present a general simple patchwork construction principle for multivariate copulas that create unfavourable VaR (i.e. Value at Risk) scenarios while maintaining given marginal distributions. This is of particular interest for the construction of Internal Models in the insurance industry under Solvency II in the European Union.

Key words: copulas, patchwork copulas, Bernstein copulas, Monte Carlo methods

AMS Classification: 62H05, 62H12, 62H17, 11K45

1. Introduction

Reasonable VaR-estimates from original data or suitable scenarios for risk management within so-called Internal Models are – besides the banking sector under Basel III – of particular interest in the insurance industry under Solvency II (see, e.g., Cadoni (2014); Cruz (2009); Doff (2011,2014); McNeil et al. (2015); Arbenz et al. (2012) or Sandström (2011)). In this paper, we propose a simple stochastic Monte Carlo algorithm on patchwork copulas for the generation of VaR scenarios that are suitable for comparison purposes in Internal Models for the calculation of solvency capital requirements. Note that the European Union (2015) concerning the implementation of Solvency II in the EU requires the consideration of such scenarios in several Articles, in particular in Article 259 on Risk Management Systems saying that insurance and reinsurance undertakings shall, where appropriate, include performance of stress tests and scenario analyses with regard to all relevant risks faced by the undertaking, in their risk-management system. The results of such analyses also have to be reported in the ORSA (Own Risk and Solvency Assessment, see e.g. Ozdemir (2015)) as described in Article 306 of the Commission Delegated Regulation. The problem is, however, that the Commission Delegated Regulation does not make any clear statements on how such stress tests or scenario analyses have to be performed.
Article 1 of the Commission Delegated Regulation defines a "scenario analysis" as an analysis of the impact of a combination of adverse events. The Monte Carlo simulation algorithm developed in this paper allows for a mathematically rigorous description how such scenarios can be generated, being flexible enough to cover also extreme situations.

2. Unfavourable patchwork copulas

Patchwork copulas in the context of risk management have been treated in detail by Arbenz et al. (2012), Cottin and Pfeifer (2014), Pfeifer (2013), Pfeifer et al. (2016, 2017, 2019) and Hummel (2018), among others. In several of the cited papers the question of an unfavourable, i.e. superadditive VaR estimate for a portfolio of aggregated risks was in particular emphasized, see also Pfeifer and Ragulina (2018). However, the construction of worst VaR scenarios in this context is quite complicated; a numerical approach to a constructive solution is e.g. given by the rearrangement algorithm (see e.g. Arbenz et al. (2012), Embrechts et al. (2013) or Mainik (2015)). From a practical point of view, simpler and yet explicit constructions for unfavourable VaR estimates by appropriate copula constructions seem to be a useful alternative. In this paper, we describe how such a construction could be performed. We start with an explicit approach in two dimensions that is later extended to arbitrary dimensions.

Theorem 1. Let, for $d \geq 2,\ d \in \mathbb{N},\ \mathbf{U} = (U_1, \cdots, U_d)$ and $\mathbf{V} = (V_1, \cdots, V_d)$ be d-dimensional random vectors over $[0,1]^d$ with continuous uniform margins (i.e., **U** and **V** represent d-dimensional copulas). Let further I denote a binomially distributed random variable, independent of **U** and **V**, with $P(I=1) = p \in (0,1)$. Then the random vector **W** with components $W_i := I \cdot p \cdot U_i + (1-I) \cdot [p + (1-p) \cdot V_i]$ for $1 \leq i \leq d$ also has continuous uniform margins, i.e. **W** represents a d-dimensional copula (a kind of patchwork copula).

Proof: The density of $p \cdot U_i$ is given by $f_i(x) = \begin{cases} \frac{1}{p}, & 0 \leq x \leq p \\ 0, & \text{otherwise} \end{cases}$ and the density of $p + (1-p) \cdot V_i$ by $g_i(x) = \begin{cases} \frac{1}{1-p}, & p \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$ which implies that the density of W_i is given by the mixture density $p \cdot f_i(x) + (1-p) \cdot g_i(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$ •

Suppose now that a portfolio of d insurance risks is considered where a mutual probabilistic dependence structure is assumed, described by **U**. If the d (for simplicity assumed continuous) marginal risk distribution functions are denoted by $F_1, \cdots, F_d$ and by $Q_1, \cdots, Q_d$ their pseudo-inverses (quantile functions), then both random vectors $(Q_1(U_1), \cdots, Q_d(U_d))$ and $(Q_1(W_1), \cdots, Q_d(W_d))$ represent a risk vector $\mathbf{X} = (X_1, \cdots, X_d)$ with the given marginal distributions. However, w.r.t. to risk aggregation, $\mathbf{X} := (Q_1(W_1), \cdots, Q_d(W_d))$ creates in general an unfavourable VaR scenario for $S=\sum_{i=1}^{d} X_i$, even if p is close to 1 and therefore **U** and **W** differ only marginally. The following graph shows the corresponding support of **W** in two dimensions.

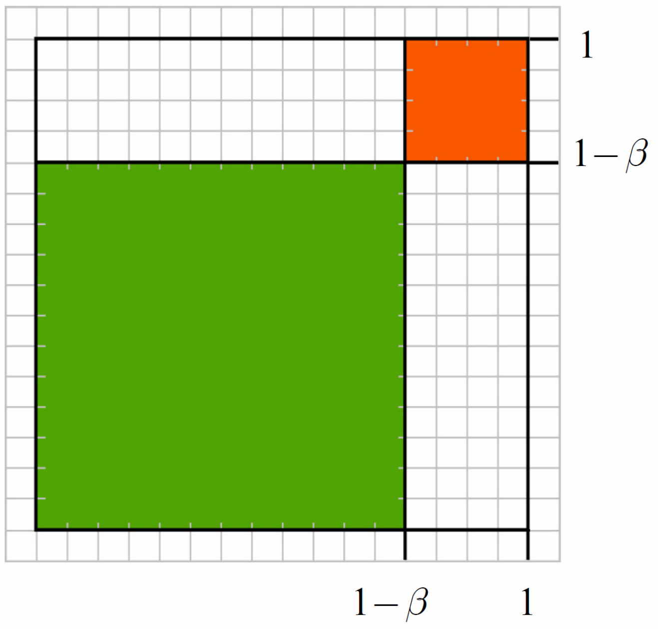


Fig. 1

In the sequel put $p:=1-\beta$ for $0<\beta<1$. Then $\mathbf{W}=I\cdot(1-\beta)\cdot\mathbf{U}+(1-I)\cdot\left(1-\beta+\beta\cdot\mathbf{V}\right)$.

We start with some preliminary Lemmata.

Lemma 1. Let $\mathbf{W}_1:=(1-\beta)\cdot\mathbf{U}$, $\mathbf{W}_2:=1-\beta+\beta\cdot\mathbf{V}$, $Z_{1i}:=Q_i\left(W_{1i}\right)$ and $Z_{2i}:=Q_i\left(W_{2i}\right)$, $i=1,2.$ Then there hold

$$F_{Z_{1i}}(x,\beta)=\begin{cases}\dfrac{F_i(x)}{1-\beta}, & 0\leq x\leq Q_i(1-\beta)\\ 1, & x\geq Q_i(1-\beta)\end{cases} \quad \text{and} \quad F_{Z_{2i}}(x,\beta)=\begin{cases}0, & 0\leq x\leq Q_i(1-\beta)\\ \dfrac{F_i(x)+\beta-1}{\beta}, & x\geq Q_i(1-\beta).\end{cases}$$

Proof. We have

$$F_{Z_{1i}}(x,\beta)=P\left(Q_i\left((1-\beta)\cdot U_i\right)\leq x\right)=P\left((1-\beta)\cdot U_i\leq F_i(x)\right)$$
$$=P\left(U_i\leq\frac{F_i(x)}{1-\beta}\right)=\frac{F_i(x)}{1-\beta},\quad 0\leq x\leq Q_i(1-\beta)$$

and

$$F_{Z_{2i}}(x,\beta) = P\left(Q_i\left(1-\beta+\beta\cdot V_i\right)\le x\right) = P\left(1-\beta+\beta\cdot V_i \le F_i(x)\right) = P\left(V_i \le \frac{F_i(x)+\beta-1}{\beta}\right)$$
$$= \frac{F_i(x)+\beta-1}{\beta},\ x \ge Q_i(1-\beta),\ i=1,2. \quad \bullet$$

Lemma 2. Assume that *f* and *g* are Lebesgue densities of independent random variables *X* and *Y*, concentrated on the same finite interval $[0,M]$ with $M>0$. Then $S := X+Y$ has the density h_1 given by

$$h_1(x) = \begin{cases} \int_0^x f(x-y)g(y)\,dy, & 0\le x\le M \\ \int_{x-M}^{M} f(x-y)g(y)\,dy, & M\le x\le 2M. \end{cases}$$

If f and g are concentrated on the same infinite interval $[M,\infty)$ with $M\ge 0$, then $S := X+Y$ has the density h_2 given by

$$h_2(x) = \int_M^{x-M} f(x-y)g(y)\,dy,\ x\ge 2M.$$

In particular, if *F* and *G* are the corresponding cdf's pertaining to *f* and *g*, resp., then in either case, $\left.\frac{d}{dx}F*G(x)\right|_{x=2M} = 0$, where $*$ means convolution.

Proof. In the finite interval case, we have, by the usual convolution formula, $h_1(x) = \int_{\substack{0\le y\le M\\ 0\le x-y\le M}} f(x-y)g(y)\,dy = \int_{\max(0,x-M)\le y\le \min(x,M)} f(x-y)g(y)\,dy.$ Now for $0\le x\le M$, we have $\max(0,x-M)=0$, $\min(x,M)=x$, from which the upper formula in brackets above follows. For $M\le x\le 2M$, we have $\max(0,x-M)=x-M$, $\min(x,M)=M$, from which the lower formula in brackets above follows.
The proof for the infinite interval case is analogous, observing that for $x\ge 2M$, we have $h_2(x) = \int_{\substack{M\le y\le x\\ M\le x-y}} f(x-y)g(y)\,dy = \int_{M\le y\le x-M} f(x-y)g(y)\,dy.$

Further, under the conditions made, we have, in either case,

$$\frac{d}{dx} F * G(x) \Big|_{x=2M} = h_1(2M) = h_2(2M) = \int_M^M f(x-y)g(y)\,dy = 0,$$

as stated. ●

Lemma 3. Assume all $F_i \equiv F$ being equal with quantile function Q, and that **U** and **V** have independent components each. Denote

$$\underline{F}(x,\beta) := \begin{cases} \dfrac{F(x)}{1-\beta}, & x \le Q(1-\beta) \\ 1, & x \ge Q(1-\beta) \end{cases} \quad \text{and} \quad \overline{F}(x,\beta) := \frac{F(x+Q(1-\beta)) + \beta - 1}{\beta},\ x \ge 0.$$

Let further denote $X_i := Q(W_i)$ and $S := \sum_{i=1}^{d} X_i$. Then we can conclude that

$$F_S(x,\beta) = \begin{cases} (1-\beta)\underline{F}^{d*}(x,\beta), & x \le dQ(1-\beta) \\ (1-\beta) + \beta \overline{F}^{d*}\left(x - dQ(1-\beta), \beta\right), & x > dQ(1-\beta), \end{cases}$$

where $*$ again means convolution. If F has a density f, then correspondingly

$$\underline{f}(x,\beta) := \begin{cases} \dfrac{f(x)}{1-\beta}, & x \le Q(1-\beta) \\ 0, & x \ge Q(1-\beta) \end{cases} \quad \text{and} \quad \overline{f}(x,\beta) := \frac{f(x+Q(1-\beta))}{\beta},\ x \ge 0$$

and

$$f_S(x,\beta) = \begin{cases} (1-\beta)\underline{f}^{d*}(x,\beta), & x \le dQ(1-\beta) \\ (1-\beta) + \beta \overline{f}^{d*}\left(x - dQ(1-\beta), \beta\right), & x > dQ(1-\beta). \end{cases}$$

Proof. Let ξ_i and ζ_i be independent random variables with the cdf's $\underline{F}(\bullet,\beta)$ and $\overline{F}(\bullet,\beta)$, resp. Then $I \cdot \xi_i + (1-I) \cdot \left(Q(1-\beta) + \zeta_i\right)$ is a stochastic representation of X_i, $i = 1, \cdots, d$, where again I is a binomial random variable with $P(I=1) = 1-\beta$ and $P(I=0) = \beta$, independent of $(\mathbf{U}, \mathbf{V})$, according to Lemma 1. Hence

$$I\cdot\sum_{i=1}^{d}\xi_i+(1-I)\cdot\sum_{i=1}^{d}\left(Q(1-\beta)+\zeta_i\right)=I\cdot\sum_{i=1}^{d}\xi_i+(1-I)\cdot\left(dQ(1-\beta)+\sum_{i=1}^{d}\zeta_i\right)$$

is a stochastic representation of *S*. Note that the cdf of $\sum_{i=1}^{d}\xi_i$ is $\underline{F}^{d*}(\bullet,\beta)$ and that of $\sum_{i=1}^{d}\zeta_i$ is $\overline{F}^{d*}(\bullet,\beta)$, from which the assertion follows. •

The following examples show the effect of a risk aggregation with an unfavourable VaR scenario for two dimensions in detail.

Example 1 (exponential distributions)**.** Assume that $F_1=F_2=\begin{cases}0, & x<0\\ 1-e^{-x}, & x\geq 0.\end{cases}$ Then

$$F_{Z_{1i}}(x,\beta)=\frac{1-e^{-x}}{1-\beta},0\leq x\leq -\ln(\beta)\text{ and }F_{Z_{2i}}(x,\beta)=\frac{\beta-e^{-x}}{\beta}=1-e^{-x-\ln(\beta)},x\geq -\ln(\beta),i=1,2.$$

For the corresponding densities, we obtain by differentiation

$$f_{Z_{1i}}(x,\beta)=\begin{cases}\dfrac{e^{-x}}{1-\beta}, & 0\leq x\leq -\ln(\beta)\\ 0, & x>-\ln(\beta)\end{cases}\quad\text{and}\quad f_{Z_{2i}}(x,\beta)=\begin{cases}0, & x<-\ln(\beta)\\ e^{-x-\ln(\beta)}, & x\geq -\ln(\beta)\end{cases},\ i=1,2$$

and

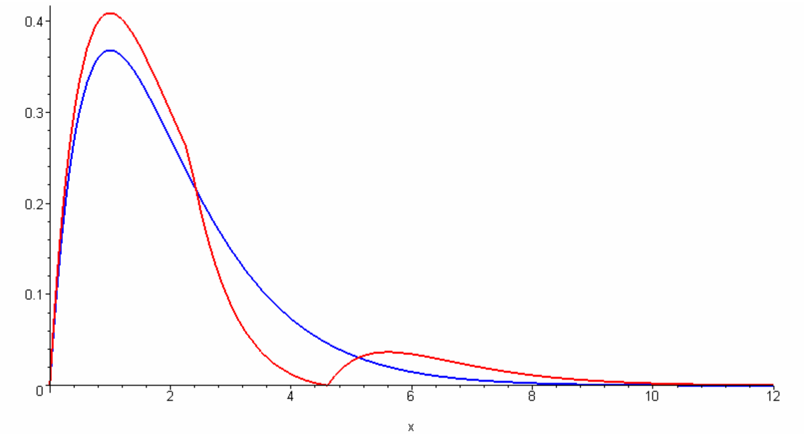
$$\underline{f}(x,\beta)=\begin{cases}\dfrac{e^{-x}}{1-\beta}, & 0\leq x\leq -\ln(\beta)\\ 0, & x>-\ln(\beta)\end{cases}\quad\text{and}\quad \overline{f}(x,\beta)=\begin{cases}0, & x<0\\ e^{-x}, & x\geq 0.\end{cases}$$

By Lemma 3, we obtain the following density f_S of the aggregated risk *S*:

$$f_S(x,\beta)=\begin{cases}\dfrac{xe^{-x}}{1-\beta}, & 0\leq x\leq -\ln(\beta)\\ \dfrac{\left(-2\ln(\beta)-x\right)e^{-x}}{1-\beta}, & -\ln(\beta)\leq x\leq -2\ln(\beta)\\ \dfrac{\left(x+2\ln(\beta)\right)e^{-x}}{\beta} & x\geq -2\ln(\beta)\end{cases}$$

with the corresponding cdf F_S :

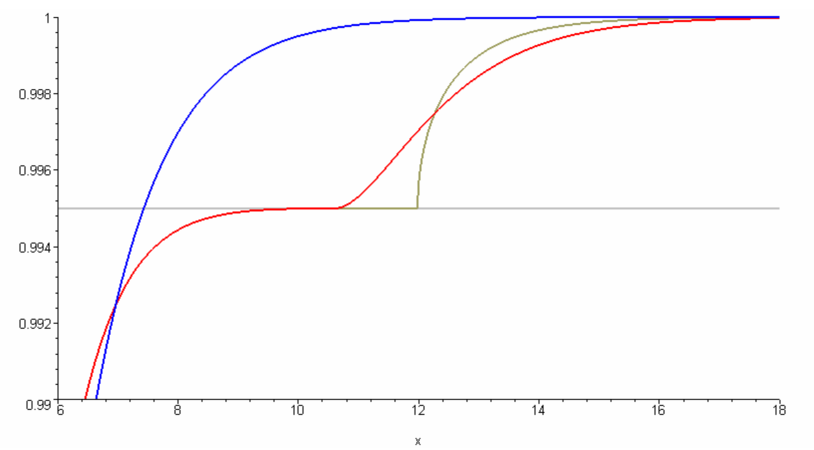
$$F_S(x,\beta)=\begin{cases}\dfrac{1-(1+x)e^{-x}}{1-\beta}, & 0\le x\le -\ln(\beta)\\[2ex] \dfrac{1-2\beta+2e^{-x}\ln(\beta)+(1+x)e^{-x}}{1-\beta}, & -\ln(\beta)\le x\le -2\ln(\beta)\\[2ex] \dfrac{\beta-2e^{-x}\ln(\beta)-(1+x)e^{-x}}{\beta} & x\ge -2\ln(\beta).\end{cases}$$



plots of $f_S(x,\beta)$ for $\beta=0.1$ (red) and $g(x)$ (blue)

Fig. 2

Here g is the density of $T:=Q_1(U_1)+Q_2(U_2)$ (independent summands, Gamma distribution).



plots of $F_S(x,\beta)$ (red), $G(x)$ (blue), and $H(x,\beta)$ (khaki), for $\beta=0.005$

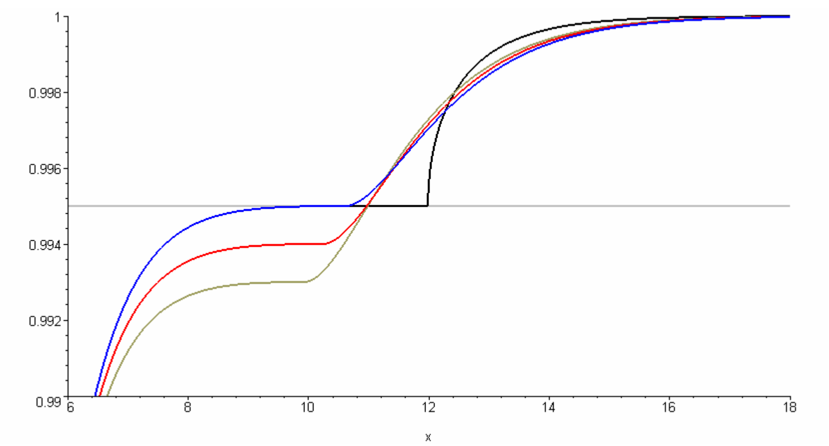
Fig. 3

Here G is the cdf for $T := Q_1(U_1) + Q_2(U_2)$ (independent summands, Gamma distribution) and H the cdf for S under the worst VaR scenario, i.e. the distribution of $\mathbf{V}$ corresponds to the lower Fréchet bound or countermonotonicity copula (see e.g. Embrechts et al. (2013) or Pfeifer (2013)). In this case we have

$$H(x,\beta) = \begin{cases} F_S(x), & x \leq -2\ln(\beta) \\ 1-\beta, & -2\ln(\beta) \leq x \leq -2\ln(\beta/2) \\ 1-\beta+\sqrt{\beta^2 - 4e^{-x}}, & x \geq -2\ln(\beta/2). \end{cases}$$

Note that with the Solvency II standard $\alpha = 0.005$, we get here, for $\beta = \alpha$, $\mathrm{VaR}_\alpha(S) = 10.5914 > \mathrm{VaR}_\alpha(T) = 7.4301$. For the worst VaR scenario, however, we get $\mathrm{wVaR}_\alpha\left(S^*\right) = 11.9829 > 10.5966 = \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)$. Note that actually the worst VaR is obtained as a limit of $\mathrm{VaR}_{\alpha+\varepsilon}\left(S^*\right)$ for $\varepsilon \downarrow 0$ due to the right continuity of cdf's. Seemingly $\mathrm{VaR}_\alpha(S) = 10.5914 < 10.5966 = \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)$ which means that even with the construction for S with $\beta = \alpha$, we still have a (quite small) diversification effect, but not in the worst VaR scenario. This changes, however, if we look at $\mathrm{VaR}_\alpha(S) = 10.9630$ when we replace β by $\alpha + \varepsilon$ in the definition of $\mathbf{W}$ for e.g. $\varepsilon = 0.001$.

The following graph shows the cdf's for several choices of ε.



plots of $F_S(x, 0.005+\varepsilon)$ for $\varepsilon = 0.001$ (blue), $\varepsilon = 0.002$ (red), $\varepsilon = 0.003$ (khaki) and $H(x, 0.005)$ (black)

Fig. 4

The following graph shows the values of $Q_S(0.995,\beta) = F_S^{-1}(0.995,\beta)$ in the range $0.0062 \leq \beta \leq 0.0076$.

A numerical calculation shows that for $\alpha = 0.005$ the worst $\mathrm{VaR}_\alpha(S) = 10.98292909$ is attained for $\beta = 0.00679331,$ i.e. $\varepsilon = 0.00179331.$

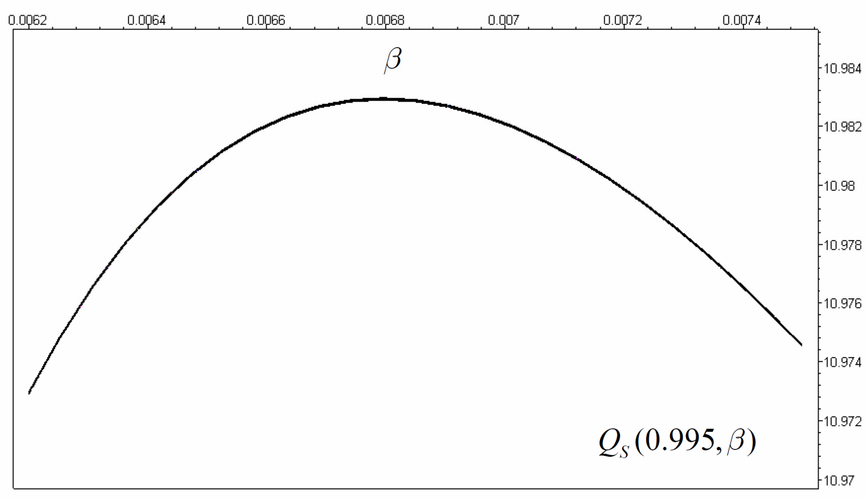


Fig. 5

Example 2 (uniform distributions). Assume that $F_1 = F_2 = \begin{cases} 0, & x \le 0 \\ x, & 0 \le x \le 1 \\ 1, & x \ge 1. \end{cases}$ Then

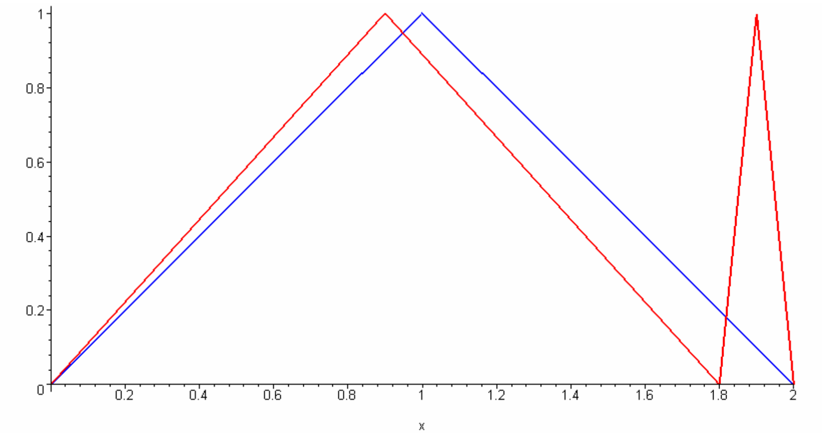
$$F_{Z_{1i}}(x,\beta) = \frac{x}{1-\beta}, \quad 0 \le x \le 1-\beta \quad \text{and} \quad F_{Z_{2i}}(x,\beta) = \frac{x+\beta-1}{\beta}, \quad x \ge 1-\beta, \; i = 1,2.$$

By Lemma 2, we obtain the following density f_S of the aggregated risk *S*:

$$f_S(x,\beta) = \begin{cases} \dfrac{x}{1-\beta}, & x \le 1-\beta \\ \dfrac{2-2\beta-x}{1-\beta}, & 1-\beta \le x \le 2-2\beta \\ \dfrac{x-2+2\beta}{\beta}, & 2-2\beta \le x \le 2-\beta \\ \dfrac{2-x}{\beta}, & 2-\beta \le x \le 2 \end{cases}$$

with the corresponding cdf F_S :

$$F_S(x,\beta)=\begin{cases} \dfrac{x^2}{2(1-\beta)}, & x\le 1-\beta \\ \dfrac{4x(1-\beta)-x^2-2(1-\beta)^2}{2(1-\beta)}, & 1-\beta\le x\le 2-2\beta \\ \dfrac{4(1-\beta)(1-x)+x^2-2\beta+2\beta^2}{2\beta}, & 2-2\beta\le x\le 2-\beta \\ \dfrac{2\beta-4(1-x)-x^2}{2\beta}, & 2-\beta\le x\le 2. \end{cases}$$

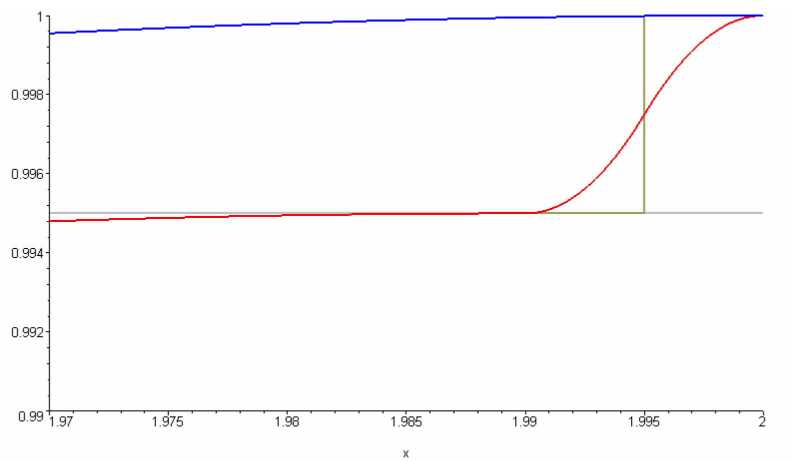


s

plots of $f_S(x,\beta)$ for $\beta=0.1$ (red) and $g(x)$ (blue)

Fig. 6

Here g is the density of $T:=Q_1(U_1)+Q_2(U_2)$ (independent summands, triangle distribution).



plots of $F_S(x,\beta)$ (red), $G(x)$ (blue), and $H(x,\beta)$ (khaki) for $\beta=0.005$

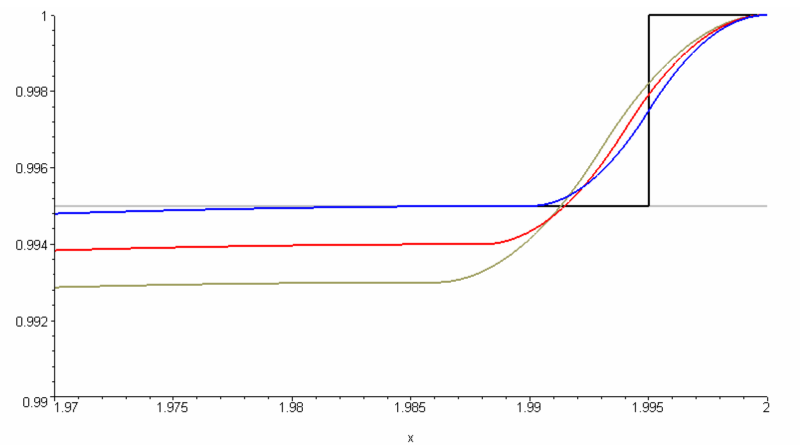
Fig. 7

Here G is the cdf for $T:=Q_1(U_1)+Q_2(U_2)$ (independent summands, triangle distribution) and H the cdf for S under the worst VaR scenario, i.e. the distribution of **V** corresponds to the upper Fréchet bound (see e.g. Embrechts et al. (2013) or Pfeifer (2013)). In this case we have

$$H(x,\beta)=\begin{cases}F_S(x) & x\le 2-2\beta\\ 1-\beta, & 2-2\beta\le x<2-\beta\\ 1, & x\ge 2-\beta.\end{cases}$$

Note that with the Solvency II standard $\alpha = 0.005$, we have here, for $\beta=\alpha$, $\mathrm{VaR}_\alpha(S)=1.99=\mathrm{VaR}_\alpha(T)$. For the worst VaR scenario, however, we get here $\mathrm{wVaR}_\alpha\left(S^*\right)=1.995>1.99=\mathrm{VaR}_\alpha(X_1)+\mathrm{VaR}_\alpha(X_2)$. Note that actually the worst VaR is obtained as a limit of $\mathrm{VaR}_{\alpha+\varepsilon}\left(S^*\right)$ for $\varepsilon\downarrow 0$ due to the right continuity of cdf's. Seemingly $\mathrm{VaR}_\alpha(S)=1.99=\mathrm{VaR}_\alpha(X_1)+\mathrm{VaR}_\alpha(X_2)$ which means that with the construction for S we have no true diversification effect, likewise in the worst VaR scenario. This changes, however, if we look at $\mathrm{VaR}_\alpha(S)=1.991$ when we replace β by $\alpha+\varepsilon$ in the definition of **W** for e.g. $\varepsilon=0.001$.

The following graph shows the cdf's for several choices of ε.



plots of $F_S(x,0.005+\varepsilon)$ for $\varepsilon=0.001$ (blue), $\varepsilon=0.002$ (red), $\varepsilon=0.003$ (khaki) and $H(x,0.005)$ (black)

Fig. 8

The following graph shows the values of $Q_S(0.995,\beta)=F_S^{-1}(0.995,\beta)$ in the range $0.0054\le\beta\le 0.007$.

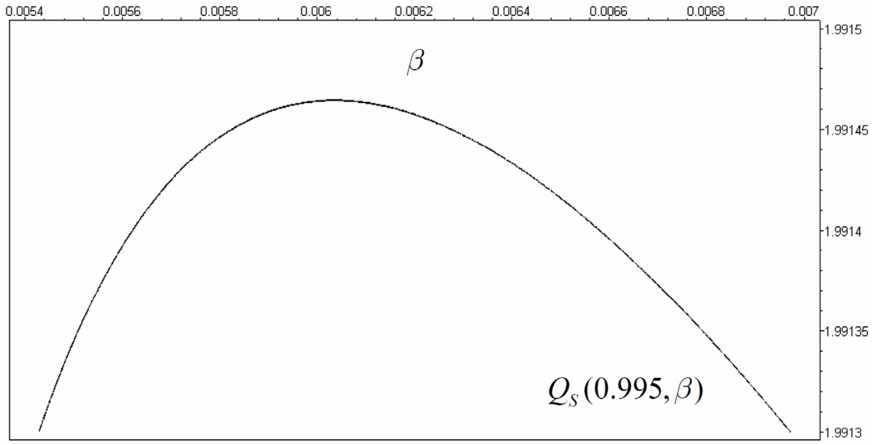


Fig. 9

A numerical calculation shows that for $\alpha = 0.005$ the worst $\text{VaR}_\alpha(S) = 1.991464466$ is attained for $\beta = 0.006035$, i.e. $\varepsilon = 0.001035$.

Note that in this example a closed-form representation for $Q_S(u,\beta)$ is given by

$Q_S(u,\beta) = 2 - 2\beta + \sqrt{2\beta(\beta + u - 1)}, 1-\beta \le u \le 1 - \frac{\beta}{2}$. This implies

$$Q_S(1-\alpha,\beta) = 2 - 2\beta + \sqrt{2\beta(\beta-\alpha)}, \alpha \le \beta \le 2\alpha$$

with its maximum being attained for $\beta_0 = \frac{1+\sqrt{2}}{2}\alpha$ with value

$Q_S(1-\alpha,\beta_0) = 2 - \left(1 + \frac{\sqrt{2}}{2}\right)\alpha$. Note that the worst VaR here is $\text{wVaR}_\alpha\left(S^*\right) = 2 - \alpha$.

Example 3 (Pareto distributions)**.** Assume that $F_1 = F_2 = \begin{cases} 0, & x \le 0 \\ \frac{x}{1+x}, & x > 0. \end{cases}$ Then

$$F_{Z_{1i}}(x,\beta) = \frac{x}{(1-\beta)(1+x)}, \quad 0 \le x \le \frac{1}{\beta} - 1 \quad \text{and} \quad F_{Z_{2i}}(x,\beta) = 1 - \frac{1}{\beta(1+x)}, \quad x \ge \frac{1}{\beta} - 1, \ i = 1,2.$$

For the corresponding densities, we obtain by differentiation

$$f_{Z_{1i}}(x,\beta) = \begin{cases} \frac{1}{(1-\beta)(1+x)^2}, & 0 \le x \le \frac{1}{\beta} - 1 \\ 0, & x > \frac{1}{\beta} - 1 \end{cases} \quad \text{and} \quad f_{Z_{2i}}(x,\beta) = \begin{cases} 0, & x < \frac{1}{\beta} - 1 \\ \frac{1}{\beta(1+x)^2}, & x \ge \frac{1}{\beta} - 1 \end{cases}$$

and

$$\underline{f}(x,\beta)=\begin{cases}\dfrac{1}{(1-\beta)(1+x)^2}, & 0\le x\le\dfrac{1}{\beta}-1\\[2ex] 0, & x>\dfrac{1}{\beta}-1\end{cases}\quad\text{and}\quad \overline{f}(x,\beta)=\begin{cases}0, & x<0\\[2ex] \dfrac{\beta}{(1+\beta x)^2} & x\ge 0.\end{cases}$$

In order to calculate the density f_S of the aggregated risk *S*, we need a suitable partial fraction representation of $\underline{f}(x-y)\underline{f}(y)$ and $\overline{f}(x-y)\overline{f}(y)$. Note that in general, we have

$$\frac{1}{(1+x-y)(1+y)}=\frac{1}{2+x}\left[\frac{1}{1+x-y}+\frac{1}{1+y}\right]$$

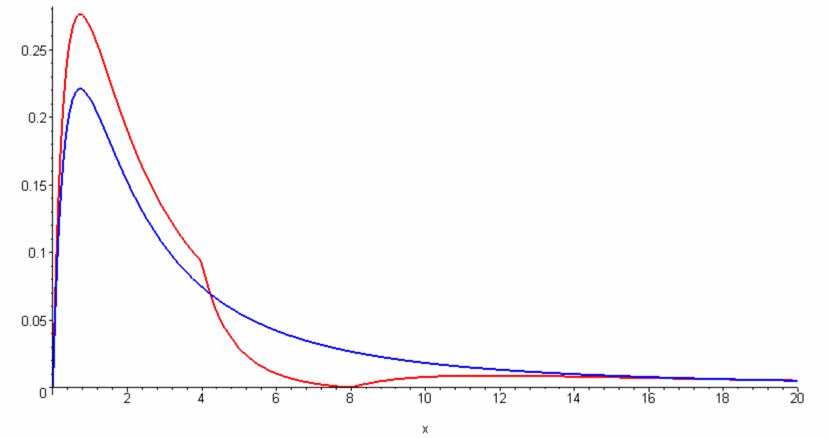
and

$$\begin{aligned}\frac{1}{(1+x-y)^2(1+y)^2}&=\frac{1}{(2+x)^2}\left[\frac{1}{(1+x-y)}+\frac{1}{(1+y)}\right]^2\\&=\frac{1}{(2+x)^2}\left[\frac{1}{(1+x-y)^2}+\frac{1}{(1+y)^2}+\frac{2}{2+x}\left[\frac{1}{1+x-y}+\frac{1}{1+y}\right]\right]\end{aligned}$$

from which we obtain, by Lemma 3,

$$F_S(x,\beta)=\begin{cases}\dfrac{x^2+2x-2\ln(1+x)}{(2+x)^2(1-\beta)}, & 0\le x\le\dfrac{1}{\beta}-1\\[2ex] \dfrac{(1-2\beta)x^2+(4-6\beta)x-4\beta+4+2\ln\left(\beta x+2\beta-1\right)}{(2+x)^2(1-\beta)}, & \dfrac{1}{\beta}-1\le x\le 2\left(\dfrac{1}{\beta}-1\right)\\[2ex] \dfrac{x^2-2x+\dfrac{2}{\beta}\ln\left(\beta x+2\beta-1\right)}{(2+x)^2} & x\ge 2\left(\dfrac{1}{\beta}-1\right).\end{cases}$$

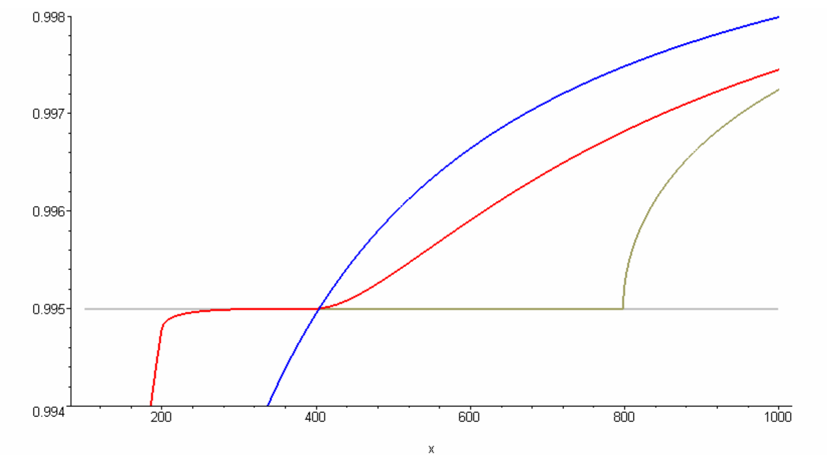
The density $f_S(x)$ follows by differentiation.



plots of $f_S(x,\beta)$ for $\beta = 0.1$ (red) and $g(x)$ (blue)

Fig. 10

Here g is the density of $T := Q_1(U_1) + Q_2(U_2)$ (independent summands).



plots of $F_S(x,\beta)$ (red), $G(x)$ (blue), and $H(x,\beta)$ (khaki) for $\beta = 0.005$

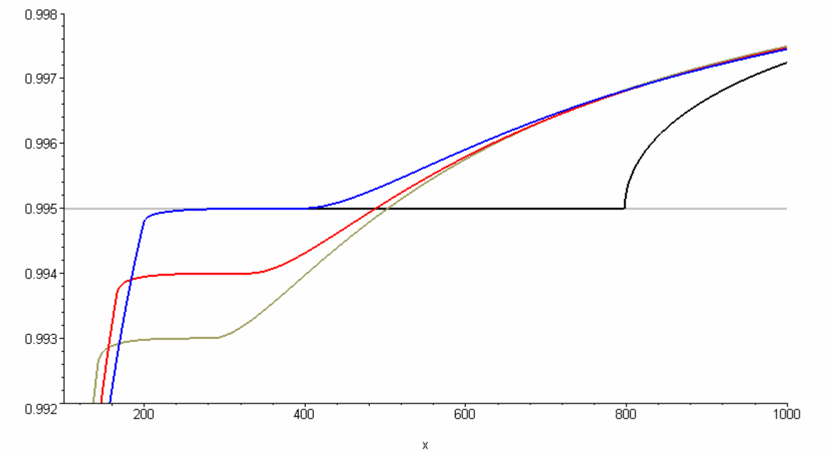
Fig. 11

Here G is the cdf for $T := Q_1(U_1) + Q_2(U_2)$ (independent summands) and H the cdf for S under the worst VaR scenario, i.e. the distribution of **V** corresponds again to the upper Fréchet bound. In this case we have

$$H(x,\beta) = \begin{cases} F_S(x,\beta), & x \le \frac{2}{\beta} - 2 \\ 1-\beta, & \frac{2}{\beta} - 2 \le x \le \frac{4}{\beta} - 2 \\ 1-\beta+\sqrt{\beta^2 - \frac{4\beta}{2+x}}, & x \ge \frac{4}{\beta} - 2. \end{cases}$$

Note that with the Solvency II standard $\alpha = 0.005$, we have here, for $\beta = \alpha$, $\text{VaR}_\alpha(S) = 397.3168 < \text{VaR}_\alpha(T) = 403.9161$. For the worst VaR scenario, however, we get $\text{wVaR}_\alpha\left(S^*\right) = 798 > 398 = \text{VaR}_\alpha(X_1) + \text{VaR}_\alpha(X_2)$. Note that actually the worst VaR is obtained as a limit of $\text{VaR}_{\alpha+\varepsilon}\left(S^*\right)$ for $\varepsilon \downarrow 0$ due to the right continuity of cdf's. Seemingly $\text{VaR}_\alpha(S) = 397.32 < 398 = \text{VaR}_\alpha(X_1) + \text{VaR}_\alpha(X_2)$ which means that even with the construction for S we still have a (quite small) diversification effect, but not in the worst VaR scenario. This changes, however, if we look at $\text{VaR}_\alpha(S) = 488.2116$ when we replace β by $\beta + \varepsilon$ in the definition of **W** for e.g. $\varepsilon = 0.001$.

The following graph shows the cdf's for several choices of ε.



plots of $F_S(x, 0.005+\varepsilon)$ for $\varepsilon = 0.001$ (blue), $\varepsilon = 0.002$ (red), $\varepsilon = 0.003$ (khaki) and $H(x, 0.005)$ (black)

Fig. 12

The following graph shows the values of $Q_S(0.995, \beta) = F_S^{-1}(0.995, \beta)$ in the range $0.007 \le \beta \le 0.012$.

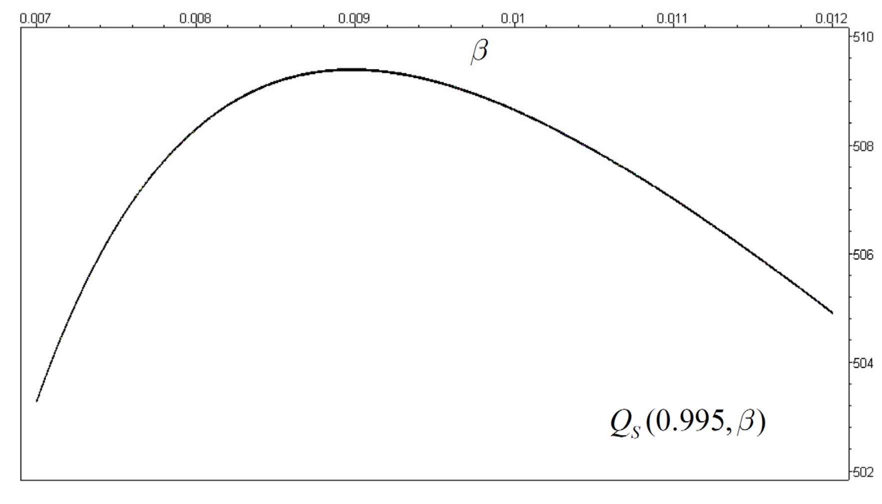


Fig. 13

A numerical calculation shows that for $\alpha = 0.005$ the worst $\mathrm{VaR}_\alpha(S) = 509.3798950$ is attained for $\beta = 0.0088963$, i.e. $\varepsilon = 0.0038963$.

These examples show that it is generally possible to obtain near worst VaR scenarios by a suitable choice of $\beta = \alpha + \varepsilon$ in the definition of **W**.

We continue with a particular construction of **W** which allows in general for an unfavourable VaR scenario.

Theorem 2. For $d \in \mathbb{N}, d > 1$ let $\mathbf{I}_d = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$ denote the d-dimensional unit matrix and $\mathbf{E}_d = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 1 \end{bmatrix}$ the $d \times d$ matrix with all entries equal to unity. Then $\Sigma_d = (1-r)\mathbf{I}_d + r\mathbf{E}_d = \begin{bmatrix} 1 & r & \cdots & r \\ r & \ddots & & \vdots \\ \vdots & & \ddots & r \\ r & \cdots & r & 1 \end{bmatrix}$ is a correlation matrix iff $-\dfrac{1}{d-1} \leq r \leq 1$. In the general case, the eigenvalues λ_i of Σ_d are given by $\lambda_1 = 1 + (d-1)r$ and

$\lambda_i = 1-r,\ i = 2,\cdots,d.$ An orthonormal basis $T_1,\cdots,T_d$ of corresponding eigenvectors is given by $T_1 = \frac{1}{\sqrt{d}}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$ and $T_j = \begin{pmatrix}t_{1j}\\ \vdots \\ t_{dj}\end{pmatrix}$ for $2 \le j \le d$ where $t_{ij} = \begin{cases} -\frac{1}{\sqrt{j(j-1)}}, & 1 \le i < j \\ \sqrt{\frac{j-1}{j}}, & j = i \\ 0, & i > j. \end{cases}$

Hence Σ_d possesses the spectral decomposition $\Sigma_d = \mathbf{A}\mathbf{A}^{tr}$ with $\mathbf{A} = \mathbf{T}\sqrt{\Delta}$ where $\mathbf{T} = [T_1,\cdots,T_d]$ and $\Delta = \begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_d\end{bmatrix}.$

To prove Theorem 2, we need the following Lemma.

Lemma 4. For all $d \ge 2,$ we have

a) $\sum_{k=2}^{d} \frac{1}{k(k-1)} = \frac{d-1}{d},$

and for $1 \le i \le d,$ we have

b) $\frac{i-1}{i} + \sum_{k=1}^{d-i} \frac{1}{(i+k)(i+k-1)} = \frac{d-1}{d}.$

Proof. Part a) of Lemma 4 can be easily proved by induction. For $d = 2,$ this is obvious. Now assume that the statement is true for some $d \ge 2.$ Then we have $\sum_{k=2}^{d+1} \frac{1}{k(k-1)} = \sum_{k=2}^{d} \frac{1}{k(k-1)} + \frac{1}{d(d+1)} = \frac{d-1}{d} + \frac{1}{d(d+1)} = \frac{d}{d+1},$ hence the statement is also true for $d+1,$ which proves a).

Part b) of Lemma 4 follows immediately from part a) since $\frac{i-1}{i} = \sum_{k=2}^{i} \frac{1}{k(k-1)}$ and $\sum_{k=1}^{d-i} \frac{1}{(i+k)(i+k-1)} = \sum_{k=i+1}^{d} \frac{1}{k(k-1)}.$ •

Proof of Theorem 2. We first show that $\mathbf{T}\mathbf{T}^{tr} = \mathbf{I}_d = \mathbf{T}^{tr}\mathbf{T}.$ Let $\mathbf{T}\mathbf{T}^{tr} = \left[b_{ij}\right]_{i,j=1,\cdots,d}.$ For $1 \le i \le d$ we obtain, by part b) of Lemma 4, $b_{ii} = \frac{1}{d} + \frac{i-1}{i} + \sum_{k=1}^{d-i} \frac{1}{(i+k)(i+k-1)} = 1.$ For

$1 \le i, j \le d$ with $i \ne j$ we get, with $i \vee j := \max(i, j)$, following part b) of Lemma 4,

$$b_{ij} = \frac{1}{d} - \frac{1}{i \vee j} + \sum_{k=i \vee j+1}^{d} \frac{1}{k(k-1)} = \frac{1}{d} - \frac{1}{i \vee j} + \sum_{k=1}^{d-i\vee j} \frac{1}{(k+i\vee j)(k+i\vee j-1)}$$

$$= \frac{1}{d} - \frac{1}{i \vee j} + \frac{d-1}{d} - \frac{i \vee j - 1}{i \vee j} = 1 - 1 = 0.$$

This proves $\mathbf{T}\mathbf{T}^{tr} = \mathbf{I}_d$. On the other hand, let $\mathbf{T}^{tr}\mathbf{T} = \left[c_{ij}\right]_{i,j=1,\cdots,d}$. It is obvious that $c_{11} = \frac{1}{d} \cdot d = 1$ and for all $2 \le i \le d$, $c_{ii} = \frac{1}{i(i-1)} \cdot (i-1) + \frac{i-1}{i} = 1.$

Next, for all $2 \le j \le d$, we obtain $c_{1j} = \frac{1}{\sqrt{d}} \left(-\frac{1}{\sqrt{j(j-1)}} \cdot (j-1) + \sqrt{\frac{j-1}{j}} \right) = 0$, and for all $2 \le i \le d$, we get $c_{i1} = \frac{1}{\sqrt{d}} \left(-\frac{1}{\sqrt{i(i-1)}} \cdot (i-1) + \sqrt{\frac{i-1}{i}} \right) = 0$. Finally, for $2 \le i, j \le d$ with $i \ne j$, we get $c_{ij} = -\frac{1}{\sqrt{(i\vee j)\cdot(i \vee j - 1)}} \cdot \left(-\frac{1}{\sqrt{(i \vee j)\cdot(i\vee j-1)}} \cdot (i \vee j - 1) + \sqrt{\frac{i \vee j - 1}{i \vee j}} \right) = 0.$

This proves $\mathbf{T}^{tr}\mathbf{T} = \mathbf{I}_d$.

Now let $\lambda_1 = 1 + (d-1)r$, $\lambda_i = 1 - r$, $i = 2, \cdots, d$ and $\Delta_t = \begin{bmatrix} \lambda_1 - t & 0 & \cdots & 0 \\ 0 & \lambda_2 - t & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_d - t \end{bmatrix}$.

A standard computation yields, for $t \in \mathbb{R}$,

$$\mathbf{T}\Delta_t = \begin{bmatrix} \frac{1+(d-1)r-t}{\sqrt{d}} & -\frac{1-r-t}{\sqrt{2(2-1)}} & -\frac{1-r-t}{\sqrt{3(3-1)}} & \cdots & -\frac{1-r-t}{\sqrt{(d-1)(d-2)}} & -\frac{1-r-t}{\sqrt{d(d-1)}} \\ \frac{1+(d-1)r-t}{\sqrt{d}} & \sqrt{\frac{2-1}{2}}(1-r-t) & -\frac{1-r-t}{\sqrt{3(3-1)}} & \cdots & -\frac{1-r-t}{\sqrt{(d-1)(d-2)}} & -\frac{1-r-t}{\sqrt{d(d-1)}} \\ \frac{1+(d-1)r-t}{\sqrt{d}} & 0 & \sqrt{\frac{3-1}{3}}(1-r-t) & \cdots & -\frac{1-r-t}{\sqrt{(d-1)(d-2)}} & -\frac{1-r-t}{\sqrt{d(d-1)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1+(d-1)r-t}{\sqrt{d}} & 0 & 0 & & \sqrt{\frac{d-2}{d-1}}(1-r-t) & -\frac{1-r-t}{\sqrt{d(d-1)}} \\ \frac{1+(d-1)r-t}{\sqrt{d}} & 0 & 0 & & 0 & \sqrt{\frac{d-1}{d}}(1-r-t) \end{bmatrix}.$$

Let $\mathbf{T}\Delta_t\mathbf{T}^{tr} = \left[d_{ij}\right]_{i,j=1,\cdots,d}$. From part a) of Lemma 4 it follows that

$$d_{11} = \frac{1+(d-1)r-t}{d} + (1-r-t)\sum_{k=2}^{d} \frac{1}{k(k-1)} = \frac{1+(d-1)r-t}{d} + (1-r-t)\cdot\frac{d-1}{d} = 1-t,$$

and for $2 \le i \le d$, part b) of Lemma 4 gives

$$d_{ii} = \frac{1+(d-1)r-t}{d} + (1-r-t)\left(\frac{i-1}{i} + \sum_{k=1}^{d-i} \frac{1}{(i+k)(i+k-1)}\right)$$
$$= \frac{1+(d-1)r-t}{d} + (1-r-t)\frac{d-1}{d} = 1-t.$$

Next, for $2 \le i, j \le d$ with $i \ne j$ we obtain from part b) of Lemma 4,

$$d_{ij} = \frac{1+(d-1)r-t}{d} - \frac{1-r-t}{i \vee j} + (1-r-t)\left(\sum_{k=1}^{d-i\vee j} \frac{1}{(i\vee j+k)(i\vee j+k-1)}\right)$$
$$= \frac{1+(\mathrm{d}-1)r-t}{d} - \frac{1-r-t}{i \vee j} + (1-r-t)\left(\frac{d-1}{d} - \frac{i\vee j-1}{i\vee j}\right) = r.$$

This in turn means $\mathbf{T}\Delta_t\mathbf{T}^{tr} = \begin{bmatrix} 1-t & r & \cdots & r \\ r & 1-t & r & \vdots \\ \vdots & r & \ddots & r \\ r & \cdots & r & 1-t \end{bmatrix} = \Sigma_d - t\mathbf{I}_d$. Consequently, the characteristic polynomial for Σ_d is given by

$$\varphi_{\Sigma_d}(t) = \det\left(\Sigma_d - t\mathbf{I}_d\right) = \det\left(\mathbf{T}\Delta_t\mathbf{T}^{tr}\right) = \det(\mathbf{T})\cdot\det\left(\Delta_t\right)\cdot\det\left(\mathbf{T}^{tr}\right) = \det(\mathbf{T})\cdot\det\left(\Delta_t\right)\cdot\det\left(\mathbf{T}^{-1}\right)$$
$$= \det\left(\Delta_t\right) = \prod_{i=1}^{d}\left(\lambda_i - t\right).$$

Hence λ_i, $1 \le i \le d$, are the eigenvalues of Σ_d. Therefore, Σ_d is a correlation matrix, i.e. positive semidefinite iff $\lambda_i \ge 0$ for all $1 \le i \le d$, i.e. $-\frac{1}{d-1} \le r \le 1$.

Thus Theorem 2 is proved. ●

In what follows we will call a Gaussian copula derived from the correlation matix $\Sigma_d = \begin{bmatrix} 1 & r & \cdots & r \\ r & \ddots & & \vdots \\ \vdots & & \ddots & r \\ r & \cdots & r & 1 \end{bmatrix}$ with $r = -\frac{1}{d-1}$ a *minimal correlation Gaussian copula*.

3. A case study

The following example shows the effects of such an approach for the 19-dimensional data set discussed in Pfeifer et al. (2019). It contains insurance losses from a non-life portfolio of natural perils in $d = 19$ areas in central Europe over a time period of 20 years. The losses are given in Mio. monetary units.

Year	Area 1	Area 2	Area 3	Area 4	Area 5	Area 6	Area 7	Area 8	Area 9	Area 10
1	23.664	154.664	40.569	14.504	10.468	7.464	22.202	17.682	12.395	18.551
2	1.080	59.545	3.297	1.344	1.859	0.477	6.107	7.196	1.436	3.720
3	21.731	31.049	55.973	5.816	14.869	20.771	3.580	14.509	17.175	87.307
4	28.99	31.052	30.328	4.709	0.717	3.530	6.032	6.512	0.682	3.115
5	53.616	62.027	57.639	1.804	2.073	4.361	46.018	22.612	1.581	11.179
6	29.95	41.722	12.964	1.127	1.063	4.873	6.571	11.966	15.676	24.263
7	3.474	14.429	10.869	0.945	2.198	1.484	4.547	2.556	0.456	1.137
8	10.02	31.283	21.116	1.663	2.153	0.932	25.163	3.222	1.581	5.477
9	5.816	14.804	128.072	0.523	0.324	0.477	3.049	7.791	4.079	7.002
10	170.725	576.767	108.361	41.599	20.253	35.412	126.698	71.079	21.762	64.582
11	21.423	50.595	4.360	0.327	1.566	64.621	5.650	1.258	0.626	3.556
12	6.38	28.316	3.740	0.442	0.736	0.470	3.406	7.859	0.894	3.591
13	124.665	33.359	14.712	0.321	0.975	2.005	3.981	4.769	2.006	1.973
14	20.165	49.948	17.658	0.595	0.548	29.35	6.782	4.873	2.921	6.394
15	78.106	41.681	13.753	0.585	0.259	0.765	7.013	9.426	2.18	3.769
16	11.067	444.712	365.351	99.366	8.856	28.654	10.589	13.621	9.589	19.485
17	6.704	81.895	14.266	0.972	0.519	0.644	8.057	18.071	5.515	13.163
18	15.55	277.643	26.564	0.788	0.225	1.230	26.800	64.538	2.637	80.711
19	10.099	18.815	9.352	2.051	1.089	6.102	2.678	4.064	2.373	2.057
20	8.492	138.708	46.708	3.68	1.132	1.698	165.6	7.926	2.972	5.237

Tab. 1

Year	Area 11	Area 12	Area 13	Area 14	Area 15	Area 16	Area 17	Area 18	Area 19
1	1.842	4.100	46.135	14.698	44.441	7.981	35.833	10.689	7.299
2	0.429	1.026	7.469	7.058	4.512	0.762	14.474	9.337	0.740
3	0.209	2.344	22.651	4.117	26.586	3.920	13.804	2.683	3.026
4	0.521	0.696	31.126	1.878	29.423	6.394	18.064	1.201	0.894
5	2.715	1.327	40.156	4.655	104.691	28.579	17.832	1.618	3.402
6	4.832	0.701	16.712	11.852	29.234	7.098	17.866	5.206	5.664
7	0.268	0.580	11.851	2.057	11.605	0.282	16.925	2.082	1.008
8	0.741	0.369	3.814	1.869	8.126	1.032	14.985	1.390	1.703
9	0.524	6.554	5.459	3.007	8.528	1.920	5.638	2.149	2.908
10	9.882	6.401	106.197	44.912	191.809	90.559	154.492	36.626	36.276
11	1.052	8.277	22.564	8.961	19.817	16.437	25.990	2.364	6.434
12	0.136	0.364	28.000	7.574	3.213	1.749	12.735	1.744	0.558
13	1.990	15.176	57.235	23.686	110.035	17.373	7.276	2.494	0.525
14	0.630	0.762	25.897	3.439	8.161	3.327	24.733	2.807	1.618
15	0.770	15.024	36.068	1.613	6.127	8.103	12.596	4.894	0.822
16	0.287	0.464	24.211	38.616	51.889	1.316	173.080	3.557	11.627
17	0.590	2.745	16.124	2.398	20.997	2.515	5.161	2.840	3.002
18	0.245	0.217	12.416	4.972	59.417	3.762	24.603	7.404	19.107
19	0.415	0.351	10.707	2.468	10.673	1.743	27.266	1.368	0.644
20	0.566	0.708	22.646	6.652	14.437	63.692	113.231	7.218	2.548

Tab. 2

A statistical analysis of the data shows a good fit to lognormal $\mathcal{LN}(\mu,\sigma)$-distributions for the losses per Area k, $k=1,\cdots,19$. The parameters μ_k and σ_k for Area k were hence estimated from the log data by calculating means and standard deviations.

Parameter	Area 1	Area 2	Area 3	Area 4	Area 5	Area 6	Area 7	Area 8	Area 9	Area 10
μ_k	2.8063	4.0717	3.1407	0.6375	0.3984	1.2227	2.3210	2.2123	1.0783	2.1055
σ_k	1.2161	1.0521	1.2110	1.5685	1.2998	1.5987	1.1980	0.9882	1.1445	1.2531

Tab. 3

Parameter	Area 11	Area 12	Area 13	Area 14	Area 15	Area 16	Area 17	Area 18	Area 19
μ_k	–0.3231	0.3815	3.0198	1.7488	3.0409	1.5501	3.0700	1.2444	0.9378
σ_k	1.0881	1.3353	0.8027	1.0033	1.1221	1.4765	0.9622	0.8577	1.2141

Tab. 4

As is to be expected, insurance losses in locally adjacent areas show a high degree of stochastic dependence, which can also be seen from the following correlation tables. For a better readability, only two decimal places are displayed.

	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12	A13	A14	A15	A16	A17	A18	A19
A1	1	0.46	0.03	0.16	0.47	0.20	0.35	0.49	0.41	0.24	0.78	0.64	0.91	0.63	0.85	0.66	0.30	0.67	0.56
A2	0.46	1	0.64	0.78	0.67	0.36	0.51	0.76	0.57	0.51	0.58	-0.04	0.59	0.84	0.68	0.58	0.87	0.77	0.90
A3	0.03	0.64	1	0.93	0.41	0.26	0.11	0.16	0.33	0.16	0.08	-0.09	0.13	0.64	0.25	0.10	0.74	0.14	0.35
A4	0.16	0.78	0.93	1	0.54	0.36	0.16	0.25	0.43	0.19	0.22	-0.10	0.30	0.79	0.36	0.19	0.84	0.32	0.49
A5	0.47	0.67	0.41	0.54	1	0.41	0.35	0.51	0.84	0.63	0.59	0.02	0.64	0.67	0.59	0.50	0.58	0.71	0.67
A6	0.20	0.36	0.26	0.36	0.41	1	0.07	0.11	0.28	0.19	0.28	0.14	0.31	0.42	0.24	0.27	0.39	0.27	0.40
A7	0.35	0.51	0.11	0.16	0.35	0.07	1	0.44	0.27	0.19	0.48	-0.07	0.46	0.35	0.45	0.91	0.64	0.61	0.49
A8	0.49	0.76	0.16	0.25	0.51	0.11	0.44	1	0.50	0.75	0.61	-0.03	0.54	0.47	0.71	0.53	0.40	0.75	0.90
A9	0.41	0.57	0.33	0.43	0.84	0.28	0.27	0.50	1	0.66	0.68	-0.01	0.52	0.60	0.50	0.41	0.46	0.65	0.63
A10	0.24	0.51	0.16	0.19	0.63	0.19	0.19	0.75	0.66	1	0.33	-0.12	0.27	0.28	0.43	0.24	0.23	0.45	0.65
A11	0.78	0.58	0.08	0.22	0.59	0.28	0.48	0.61	0.68	0.33	1	0.19	0.79	0.65	0.80	0.73	0.43	0.84	0.74
A12	0.64	-0.04	-0.09	-0.10	0.02	0.14	-0.07	-0.03	-0.01	-0.12	0.19	1	0.44	0.21	0.28	0.17	-0.12	0.13	0.03
A13	0.91	0.59	0.13	0.30	0.64	0.31	0.46	0.54	0.52	0.27	0.79	0.44	1	0.71	0.86	0.74	0.47	0.76	0.65
A14	0.63	0.84	0.64	0.79	0.67	0.42	0.35	0.47	0.60	0.28	0.65	0.21	0.71	1	0.74	0.54	0.79	0.68	0.72
A15	0.85	0.68	0.25	0.36	0.59	0.24	0.45	0.71	0.50	0.43	0.80	0.28	0.86	0.74	1	0.69	0.47	0.71	0.75
A16	0.66	0.58	0.10	0.19	0.50	0.27	0.91	0.53	0.41	0.24	0.73	0.17	0.74	0.54	0.69	1	0.63	0.77	0.64
A17	0.30	0.87	0.74	0.84	0.58	0.39	0.64	0.40	0.46	0.23	0.43	-0.12	0.47	0.79	0.47	0.63	1	0.59	0.64
A18	0.67	0.77	0.14	0.32	0.71	0.27	0.61	0.75	0.65	0.45	0.84	0.13	0.76	0.68	0.71	0.77	0.59	1	0.86
A19	0.56	0.90	0.35	0.49	0.67	0.40	0.49	0.90	0.63	0.65	0.74	0.03	0.65	0.72	0.75	0.64	0.64	0.86	1

correlations between original losses in adjacent areas

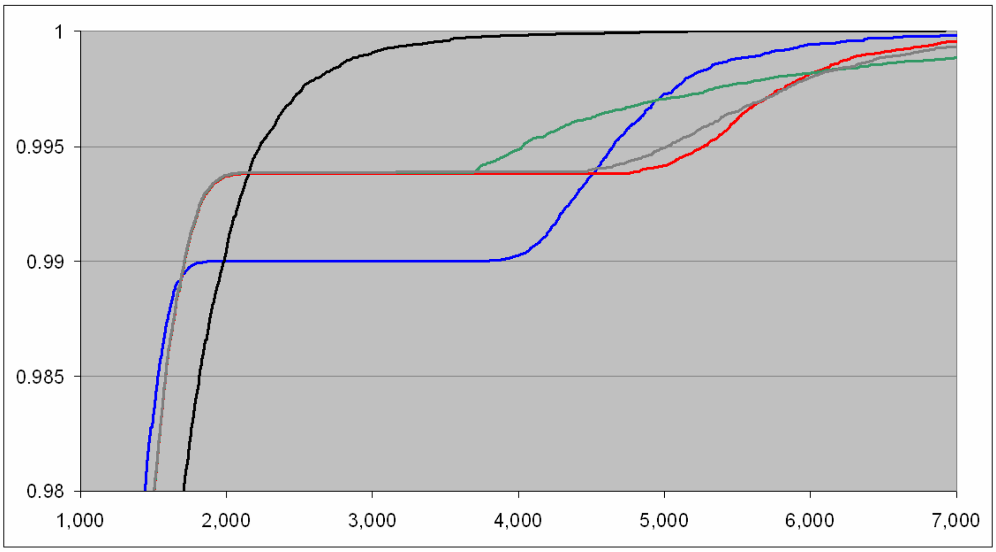
Tab. 5

	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12	A13	A14	A15	A16	A17	A18	A19
A1	1	0.27	0.30	0.16	0.17	0.45	0.28	0.32	0.32	0.29	0.67	0.51	0.76	0.34	0.67	0.74	0.18	0.21	0.29
A2	0.27	1	0.48	0.66	0.39	0.37	0.71	0.69	0.52	0.64	0.30	-0.02	0.45	0.66	0.58	0.45	0.73	0.74	0.78
A3	0.30	0.48	1	0.70	0.40	0.31	0.42	0.51	0.58	0.53	0.18	0.07	0.21	0.32	0.54	0.26	0.47	0.21	0.57
A4	0.16	0.66	0.70	1	0.77	0.47	0.46	0.47	0.59	0.49	0.18	-0.13	0.33	0.50	0.47	0.18	0.76	0.43	0.54
A5	0.17	0.39	0.40	0.77	1	0.59	0.30	0.20	0.49	0.39	0.28	0.08	0.35	0.56	0.44	0.16	0.55	0.36	0.41
A6	0.45	0.37	0.31	0.47	0.59	1	0.14	0.01	0.36	0.34	0.33	0.12	0.48	0.46	0.48	0.37	0.59	0.17	0.50
A7	0.28	0.71	0.42	0.46	0.30	0.14	1	0.52	0.27	0.40	0.45	-0.07	0.31	0.31	0.46	0.62	0.63	0.58	0.57
A8	0.32	0.69	0.51	0.47	0.20	0.01	0.52	1	0.64	0.81	0.27	-0.02	0.38	0.35	0.56	0.35	0.28	0.62	0.63
A9	0.32	0.52	0.58	0.59	0.49	0.36	0.27	0.64	1	0.78	0.40	0.19	0.27	0.50	0.44	0.30	0.33	0.57	0.61
A10	0.29	0.64	0.53	0.49	0.39	0.34	0.40	0.81	0.78	1	0.21	-0.02	0.21	0.37	0.52	0.30	0.31	0.53	0.81
A11	0.67	0.30	0.18	0.18	0.28	0.33	0.45	0.27	0.40	0.21	1	0.47	0.49	0.45	0.60	0.67	0.20	0.45	0.39
A12	0.51	-0.02	0.07	-0.13	0.08	0.12	-0.07	-0.02	0.19	-0.02	0.47	1	0.44	0.21	0.24	0.46	-0.23	0.25	0.05
A13	0.76	0.45	0.21	0.33	0.35	0.48	0.31	0.38	0.27	0.21	0.49	0.44	1	0.55	0.60	0.71	0.37	0.39	0.24
A14	0.34	0.66	0.32	0.50	0.56	0.46	0.31	0.35	0.50	0.37	0.45	0.21	0.55	1	0.59	0.43	0.57	0.58	0.53
A15	0.67	0.58	0.54	0.47	0.44	0.48	0.46	0.56	0.44	0.52	0.60	0.24	0.60	0.59	1	0.59	0.36	0.35	0.63
A16	0.74	0.45	0.26	0.18	0.16	0.37	0.62	0.35	0.30	0.30	0.67	0.46	0.71	0.43	0.59	1	0.38	0.43	0.39
A17	0.18	0.73	0.47	0.76	0.55	0.59	0.63	0.28	0.33	0.31	0.20	-0.23	0.37	0.57	0.36	0.38	1	0.52	0.56
A18	0.21	0.74	0.21	0.43	0.36	0.17	0.58	0.62	0.57	0.53	0.45	0.25	0.39	0.58	0.35	0.43	0.52	1	0.60
A19	0.29	0.78	0.57	0.54	0.41	0.50	0.57	0.63	0.61	0.81	0.39	0.05	0.24	0.53	0.63	0.39	0.56	0.60	1

correlations between log losses in adjacent areas

Tab. 6

The following graph shows estimated cdf's on a basis of 100,000 Monte Carlo simulations for the aggregated loss with a Bernstein copula representing **U** and a minimal correlation Gaussian copula representing **V**, for various values of p. For comparison purposes, we have also added an estimated cdf for the aggregated loss for a Bernstein copula representing **U** and an upper Fréchet (or comonotonicity) copula representing **V**.



plots of estimated cdf's in the tail

Fig. 14

The following graphs correspond to a Bernstein copula **U** with a minimal correlation Gaussian copula **V**: black: $p=1$; blue: $p=0.99$; red: $p=0.994$

The following graphs correspond to a Bernstein copula **U** with $p=0.994$ but different copulas **V**: green: upper Fréchet copula; grey: independence copula

The following table shows the estimated risk measures VaR_α for $\alpha=0.005$ (Solvency II-standard) for the various values of p and different types of **V**.

p	0.99	0.994	0.994	0.994	1
V	min corr Gauss	min corr Gauss	upper Fréchet	independence	---
VaR_α	4,647	5,272	4,025	5,018	2,229

Tab. 7

As can clearly be seen, the patchwork construction with the minimal correlation Gaussian copula representing **V** with no tail dependence gives the largest VaR estimate here and is typically larger than the upper Fréchet copula which has a positive tail dependence. Note that the sum of individual VaR's is given by 2,745 which means that using the Bernstein copula alone would lead to a diversified portfolio while all others do not.

Finally, it should be pointed out that the effects described here are independent of the particular copula chosen for **U**, i.e. the magnitude of the estimated VaR's under the patchwork construction would remain roughly equal also under an elliptical, an Archimedean, a vine or an adapted Bernstein copula approach for **U** (see e.g. Pfeifer and Ragulina (2020)).

References

[1] P. Arbenz, Ch. Hummel and G. Mainik (2012): Copula based hierarchical risk aggregation through sample reordering. Insurance: Mathematics and Economics 51 (2012) 122 – 133.

[2] P. Cadoni (Ed.) (2014): Internal Models and Solvency II. From Regulation to Implementation. RISK Books, London.

[3] C. Cottin and D. Pfeifer (2014): From Bernstein polynomials to Bernstein copulas. J. Appl. Funct. Anal. 9(3-4), 277 – 288.

[4] M. Cruz (Ed.) (2009): The Solvency II Handbook. Developing ERM Frameworks in Insurance and Reinsurance Companies. RISK Books, London.

[5] R. Doff (2011): Risk Management for Insurers. Risk Control, Economic Capital and Solvency II (2nd Ed.). RISK Books, London.

[6] R. Doff (Ed.) (2014): The Solvency II Handbook. Practical Approaches to Implementation. RISK Books, London.

[7] P. Embrechts, G. Puccetti, and L. Rüschendorf (2013): Model uncertainty and VaR aggregation. Journal of Banking and Finance 37: 2750 – 2764.

[8] European Union (2015): Commission Delegated Regulation EU 2015/35 of 10 October 2014 supplementing Directive 2009/138/EC of the European Parliament and of the Council on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II). Official Journal of the European Union: 17.1: L12/1–L12/797.

[9] Ch. Hummel (2018): Shaping tail dependencies by nesting box copulas. Revised version. arXiv:0906.4853v2.

[10] G. Mainik (2015): Risk aggregation with empirical margins: Latin hypercubes, empirical copulas, and convergence of sum distributions. J. Multivar. Analysis 141: 197–216.

[11] A. J. McNeil, R. Frey and P Embrechts (2015): Quantitative Risk Management. Concepts, Techniques and Tools, 2nd ed. Princeton: Princeton University Press.

[12] B. Ozdemir (2015): ORSA: Design and Implementation. RISK Books, London.

[13] D. Pfeifer (2013): Correlation, tail dependence and diversification. In: C. Becker, R. Fried, S. Kuhnt (Eds.): Robustness and Complex Data Structures. Festschrift in Honour of Ursula Gather, 301 – 314, Springer, Berlin.

[14] D. Pfeifer, H.A. Tsatedem, A. Mändle and C. Girschig (2016): New copulas based on general partitions-of-unity and their applications to risk management. Depend. Model. 4, 123 – 140.

[15] D. Pfeifer, A. Mändle and O. Ragulina (2017): New copulas based on general partitions-of-unity and their applications to risk management (part II). Depend. Model. 5, 246 – 255.

[16] D. Pfeifer and O. Ragulina (2018): Generating VaR Scenarios under Solvency II with Product Beta Distributions. Risks 6(4), 122.

[17] D. Pfeifer, A. Mändle O. Ragulina and C. Girschig (2019): New copulas based on general partitions-of-unity (part III) – the continuous case. Depend. Model. 7, 181 – 201.

[18] D. Pfeifer and O. Ragulina (2020): Adaptive Bernstein copulas and Risk Management. Preprint, arXiv: 2011.00909 [q-fin.RM]

[19] A. Sandström (2011): Handbook of Solvency for Actuaries and Risk Managers. Theory and Practice. CRC Press, Taylor & Francis Group, London.