Design of Broadband Beamformers Robust Against Gain and Phase Errors in the Microphone Array Characteristics

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***Abstract*—Fixed broadband beamformers using small-size microphone arrays are known to be highly sensitive to errors in the microphone array characteristics. This paper describes two design procedures for designing broadband beamformers with an arbitrary spatial directivity pattern, which are robust against gain and phase errors in the microphone array characteristics. The first design procedure optimizes the mean performance of the broadband beamformer and requires knowledge of the gain and the phase probability density functions, whereas the second design procedure optimizes the worst-case performance by using a minimax criterion. Simulations with a small-size microphone array show the performance improvement that can be obtained by using a robust broadband beamformer design procedure.**

***Index Terms*—Broadband beamformer, microphone characteristics, minimax, probability density function, robustness.**

I. Introduction

In MANY speech communication applications, such as hands-free mobile telephony, hearing aids, and voice-controlled systems, the recorded microphone signals are corrupted by acoustic background noise and reverberation [1]–[3]. Background noise and reverberation cause a signal degradation, which can lead to total unintelligibility of the speech and which decreases the performance of speech recognition and speech coding systems. Therefore, efficient signal enhancement algorithms are required.

Well-known multimicrophone signal enhancement techniques are fixed and adaptive beamforming [4]. Adaptive beamforming techniques, such as the generalized sidelobe canceller (GSC) and its variants [5]–[8], generally have a better noise reduction performance than fixed beamforming techniques and are able to adapt to changing acoustic environments. However, fixed beamforming techniques (with a fixed spatial directivity pattern) are sometimes preferred because they do not require a control algorithm in order to avoid signal distortion and signal cancellation and because of their easy implementation and low computational complexity. Fixed beamformers are frequently used for creating the speech and noise reference signal in a GSC, for creating multiple beams [9], [10], in applications where the position of the speech source is assumed to be (approximately) known, such as hearing aid applications [11]–[13], and in highly reverberant acoustic environments.

In this paper, we are interested in designing *robust broadband beamformers* with a given *arbitrary spatial directivity pattern* for a given *arbitrary microphone array configuration*, using an FIR *filter-and-sum* structure. Using traditional types of fixed beamformers, such as delay-and-sum beamforming, differential microphone arrays [14], superdirective microphone arrays [12], [15], [16], and frequency-invariant beamforming [17], it is generally not possible to design arbitrary spatial directivity patterns for arbitrary microphone array configurations. However, in [18] and [19], several procedures are described for designing broadband beamformers with an arbitrary spatial directivity pattern. The design consists of calculating the filter coefficients such that the spatial directivity pattern optimally fits the desired spatial directivity pattern with respect to some cost function. Different techniques can be used, e.g., weighted least-squares filter design, nonlinear optimization techniques [20]–[23], a maximum energy array [24] or eigenfilters [19], [25]. Many such broadband beamformer design procedures perform the design individually for separate frequencies and/or approximate the (double) integrals that arise in the design by a finite sum over a grid of frequencies and angles. In this paper, we will calculate full integrals over the frequency-angle plane and, hence, perform a true broadband design.

It is well known that fixed and adaptive beamformers are highly *sensitive to errors in the microphone array characteristics* (gain, phase, microphone position), especially for small-size microphone arrays. In many applications, the microphone array characteristics are not exactly known and can even change over time [26]. For superdirective beamformers, robustness against random errors can be improved by limiting the white noise gain (WNG) of the beamformer, i.e., imposing a norm constraint or a general quadratic constraint on the filter coefficients [12], [15], [16], [27]. Limiting the WNG has also been applied in order to enhance the robustness of adaptive beamformers [28]. Another

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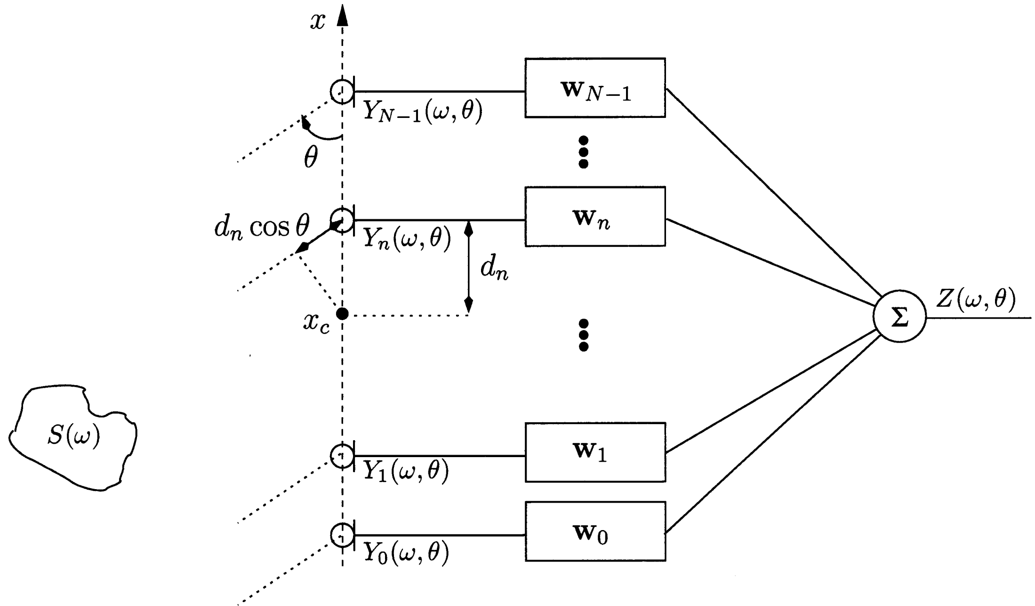

Fig. 1. Linear microphone array configuration (far-field assumption).

possibility is to perform a measurement or a calibration procedure for the used microphone array, which will, however, only limit the error sensitivity for the specific microphone array used [29], [30].

This paper discusses the design of broadband beamformers with an arbitrary spatial directivity pattern, which are robust against unknown gain and phase errors in the microphone array characteristics. In Section II, the far-field broadband beamforming problem is introduced, and some definitions and notational conventions are given. Section III discusses several cost functions that can be used for designing broadband beamformers: the weighted least-squares cost function, the eigenfilter cost function based on a total least-squares error criterion, and a nonlinear cost function. For all considered cost functions, we first discuss the general design procedure for an arbitrary spatial directivity pattern and for frequency- and angle-dependent microphone characteristics. Next, the microphone characteristics are assumed to be independent of frequency and angle, and we focus on the specific design case of a broadband beamformer having a passband and a stopband region. Using the considered cost functions, it is possible to design broadband beamformers when the microphone characteristics are exactly known. However, in many applications, the microphone characteristics are not known and can even change over time. Section IV describes two procedures for designing broadband beamformers that are robust against (unknown) gain and phase errors in the microphone array characteristics. The first design procedure optimizes the mean performance of the broadband beamformer for all feasible microphone characteristics, whereas the second design procedure optimizes the worst-case performance. Both design procedures can be used with the discussed—and other—cost functions. In Section V, simulation results for the different design procedures and cost functions are presented. It is shown that robust broadband beamformer design leads to a significant performance improvement when gain and phase errors occur.

II. Far-Field Broadband Beamforming: Configuration

Consider the linear microphone array depicted in Fig. 1, with N microphones and d_n as the distance between the nth microphone and the center of the microphone array. The *spatial directivity pattern* $H(\omega, \theta)$ for a source $S(\omega)$ with normalized frequency ω at an angle θ from the array is defined as

$$H(\omega, \theta) = \frac{Z(\omega, \theta)}{Y_c(\omega, \theta)} = \frac{\sum_{n=0}^{N-1} W_n(\omega) Y_n(\omega, \theta)}{Y_c(\omega, \theta)} \tag{1}$$

where $Y_c(\omega, \theta)$ is the received signal at the center of the microphone array, and $W_n(\omega)$ is the frequency response of the real-valued L-dimensional FIR filter $\mathbf{w}_n$

$$W_n(\omega) = \sum_{k=0}^{L-1} w_n(k) e^{-jk\omega} = \mathbf{w}_n^T \mathbf{e}(\omega) \tag{2}$$

where

$$\mathbf{w}_n = \begin{bmatrix} w_n(0) \\ w_n(1) \\ \vdots \\ w_n(L-1) \end{bmatrix} \qquad \mathbf{e}(\omega) = \begin{bmatrix} 1 \\ e^{-j\omega} \\ \vdots \\ e^{-j(L-1)\omega} \end{bmatrix}. \tag{3}$$

When the signal source is far enough from the microphone array, the far-field assumptions are valid [31], i.e., the wavefronts can be assumed to be planar, and all microphone signals can be assumed to be equally attenuated.[1] Since the microphones are not necessarily omni-directional microphones with a flat frequency response, the microphone characteristics have to be taken into account. The microphone characteristics of the nth microphone are described by the function

$$A_n(\omega, \theta) = a_n(\omega, \theta) e^{-j\gamma_n(\omega, \theta)} \tag{4}$$

[1]Since we consider small-size microphone arrays in this paper, the far-field assumption will generally be valid. However, all expressions can be easily extended to the near-field case [18], [19].

where both the gain $a_n(\omega, \theta)$ and the phase $\gamma_n(\omega, \theta)$ can be frequency-and angle-dependent. The microphone signals $Y_n(\omega, \theta)$, $n = 0 \cdots N-1$ phase-shifted and filtered versions of the signal $Y_c(\omega, \theta)$

$$Y_n(\omega, \theta) = A_n(\omega, \theta) e^{-j\omega\tau_n(\theta)} Y_c(\omega, \theta) \quad -\pi \le \omega \le \pi, \ -\pi \le \theta \le \pi \tag{5}$$

with the delay $\tau_n(\theta)$ in number of samples equal to

$$\tau_n(\theta) = \frac{d_n \cos\theta}{c} f_s \tag{6}$$

where c is the speed of sound propagation [c =340 (m/s)], and f_s is the sampling frequency. Combining (1) and (5), the **spatial directivity pattern** $H(\omega, \theta)$ can be written as

$$\boxed{H(\omega,\theta) = \sum_{n=0}^{N-1} W_n(\omega) A_n(\omega,\theta) e^{-j\omega\tau_n(\theta)} = \mathbf{w}^T \overline{\mathbf{g}}(\omega,\theta)} \tag{7}$$

with the real-valued M-dimensional filter vector $\mathbf{w}$, with $M = LN$, and the steering vector $\overline{\mathbf{g}}(\omega, \theta)$ equal to

$$\mathbf{w} = \begin{bmatrix} \mathbf{w}_0 \\ \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_{N-1} \end{bmatrix}$$

$$\overline{\mathbf{g}}(\omega, \theta) = \begin{bmatrix} \mathbf{e}(\omega) A_0(\omega, \theta) e^{-j\omega\tau_0(\theta)} \\ \mathbf{e}(\omega) A_1(\omega, \theta) e^{-j\omega\tau_1(\theta)} \\ \vdots \\ \mathbf{e}(\omega) A_{N-1}(\omega, \theta) e^{-j\omega\tau_{N-1}(\theta)} \end{bmatrix}. \tag{8}$$

The steering vector $\overline{\mathbf{g}}(\omega, \theta)$ can be written as

$$\overline{\mathbf{g}}(\omega, \theta) = \mathbf{A}(\omega, \theta) \cdot \mathbf{g}(\omega, \theta) \tag{9}$$

where $\mathbf{A}(\omega, \theta)$ is an $M \times M$-dimensional diagonal matrix consisting of the microphone characteristics, and hence, $\mathbf{g}(\omega, \theta)$ is the steering vector assuming omni-directional microphones with a flat frequency response equal to 1, i.e., $A_n(\omega, \theta) = 1$, $n = 0 \cdots N-1$

$$\mathbf{A}(\omega,\theta) = \begin{bmatrix} A_0(\omega,\theta)\mathbf{I}_L & & & \\ & A_1(\omega,\theta)\mathbf{I}_L & & \\ & & \ddots & \\ & & & A_{N-1}(\omega,\theta)\mathbf{I}_L \end{bmatrix}$$

$$\mathbf{g}(\omega,\theta) = \begin{bmatrix} \mathbf{e}(\omega) e^{-j\omega\tau_0(\theta)} \\ \mathbf{e}(\omega) e^{-j\omega\tau_1(\theta)} \\ \vdots \\ \mathbf{e}(\omega) e^{-j\omega\tau_{N-1}(\theta)} \end{bmatrix} \tag{10}$$

where $\mathbf{I}_L$ is the $L \times L$-dimensional identity matrix. The ith element of $\overline{\mathbf{g}}(\omega, \theta)$ is equal to

$$\begin{aligned} \overline{\mathbf{g}}^{\,i}(\omega, \theta) &= a_n(\omega, \theta) e^{-j\gamma_n(\omega,\theta)} \mathbf{g}^i(\omega, \theta) \\ &= a_n(\omega, \theta) \exp\left(-j \left[\omega \left(k + \frac{d_n \cos\theta}{c} f_s\right) + \gamma_n(\omega, \theta)\right]\right) \end{aligned} \tag{11}$$

where $k = \text{mod}(i-1, L)$ and $n = \lfloor (i-1)/L \rfloor$, where $\lfloor (i-1)/L \rfloor$ denotes the largest integer smaller than or equal to $(i-1)/L$, and $\text{mod}(i-1, L)$ is the remainder of the division. The steering vector $\overline{\mathbf{g}}(\omega, \theta)$ can be decomposed into a real and an imaginary part $\overline{\mathbf{g}}(\omega, \theta) = \overline{\mathbf{g}}_R(\omega, \theta) + j\overline{\mathbf{g}}_I(\omega, \theta)$. The real part $\overline{\mathbf{g}}_R(\omega, \theta)$ is equal to

$$\overline{\mathbf{g}}_R(\omega, \theta) = \mathbf{A}_R(\omega, \theta)\mathbf{g}_R(\omega, \theta) - \mathbf{A}_I(\omega, \theta)\mathbf{g}_I(\omega, \theta) \tag{12}$$

where $\mathbf{A}_R(\omega, \theta)$ and $\mathbf{A}_I(\omega, \theta)$ are the real and the imaginary parts of $\mathbf{A}(\omega, \theta)$, and $\mathbf{g}_R(\omega, \theta)$ and $\mathbf{g}_I(\omega, \theta)$ are the real and the imaginary parts of $\mathbf{g}(\omega, \theta)$.

Using (7), the **spatial directivity spectrum** $|H(\omega, \theta)|^2$ can be written as

$$\boxed{|H(\omega, \theta)|^2 = H(\omega, \theta) H^*(\omega, \theta) = \mathbf{w}^T \overline{\mathbf{G}}(\omega, \theta) \mathbf{w}} \tag{13}$$

where $\overline{\mathbf{G}}(\omega, \theta) = \overline{\mathbf{g}}(\omega, \theta)\overline{\mathbf{g}}^H(\omega, \theta)$, which can be written, using (9), as

$$\overline{\mathbf{G}}(\omega, \theta) = \mathbf{A}(\omega, \theta) \cdot \mathbf{G}(\omega, \theta) \cdot \mathbf{A}^H(\omega, \theta) \tag{14}$$

where $\mathbf{G}(\omega, \theta) = \mathbf{g}(\omega, \theta)\mathbf{g}^H(\omega, \theta)$. The (i, j)th element of $\overline{\mathbf{G}}(\omega, \theta)$ is equal to

$$\begin{aligned} \overline{\mathbf{G}}^{\,ij}(\omega,\theta) &= a_n(\omega,\theta) a_m(\omega,\theta) e^{-j(\gamma_n(\omega,\theta) - \gamma_m(\omega,\theta))} \mathbf{G}^{ij}(\omega,\theta) \\ &= a_n(\omega,\theta) a_m(\omega,\theta) \\ &\quad \cdot \exp\left(-j \left[\omega \left((k-l) + \frac{(d_n - d_m)\cos\theta}{c} f_s\right) \right.\right. \\ &\qquad \left.\left. + (\gamma_n(\omega,\theta) - \gamma_m(\omega,\theta))\right]\right) \end{aligned} \tag{16}$$

where $k = \text{mod}(i-1, L)$, $l = \text{mod}(j-1, L)$, $n = \lfloor (i-1)/L \rfloor$, and $m = \lfloor (j-1)/L \rfloor$. The matrix $\overline{\mathbf{G}}(\omega, \theta)$ can be decomposed into a real and an imaginary part $\overline{\mathbf{G}}(\omega, \theta) = \overline{\mathbf{G}}_R(\omega, \theta) + j\overline{\mathbf{G}}_I(\omega, \theta)$. Since the imaginary part $\overline{\mathbf{G}}_I(\omega, \theta)$ is anti-symmetric, i.e., $\mathbf{w}^T\overline{\mathbf{G}}_I(\omega, \theta)\mathbf{w} = 0$, the spatial directivity spectrum $|H(\omega, \theta)|^2$ is equal to

$$\boxed{|H(\omega,\theta)|^2 = \mathbf{w}^T \overline{\mathbf{G}}_R(\omega,\theta) \mathbf{w}}. \tag{17}$$

The real part $\overline{\mathbf{G}}_R(\omega, \theta)$ is equal to

$$\begin{aligned} \overline{\mathbf{G}}_R(\omega, \theta) &= \mathbf{A}_R(\omega, \theta)\mathbf{G}_R(\omega, \theta)\mathbf{A}_R(\omega, \theta) \\ &\quad + \mathbf{A}_I(\omega, \theta)\mathbf{G}_R(\omega, \theta)\mathbf{A}_I(\omega, \theta) \\ &\quad - \mathbf{A}_I(\omega, \theta)\mathbf{G}_I(\omega, \theta)\mathbf{A}_R(\omega, \theta) \\ &\quad + \mathbf{A}_R(\omega, \theta)\mathbf{G}_I(\omega, \theta)\mathbf{A}_I(\omega, \theta) \end{aligned} \tag{18}$$

where $\mathbf{G}_R(\omega, \theta)$ and $\mathbf{G}_I(\omega, \theta)$ are the real and the imaginary parts of $\mathbf{G}(\omega, \theta)$.

III. Broadband Beamforming Cost Functions

In this section, we discuss the design of broadband beamformers when the microphone characteristics $A_n(\omega, \theta)$ are exactly known. The design of a broadband beamformer consists of calculating the filter $\mathbf{w}$, such that $H(\omega, \theta)$ optimally fits the desired spatial directivity pattern $D(\omega, \theta)$, where $D(\omega, \theta)$ is an arbitrary two-dimensional (2-D) function in ω and θ. Several design procedures exist, depending on the specific cost function which is optimized. In this section, three different cost functions are considered:

- a weighted least-squares (LS) cost function J_{LS}, minimizing the weighted LS error between the spatial directivity pattern $H(\omega, \theta)$ and the desired spatial directivity pattern $D(\omega, \theta)$ [this cost function can be written as a quadratic function (cf. Section III-A)];
- the total least-squares (TLS) eigenfilter cost function J_{TLS}, minimizing the TLS error between the spatial directivity pattern $H(\omega, \theta)$ and the desired spatial directivity pattern $D(\omega, \theta)$ [this cost function leads to a generalized eigenvalue problem (cf. Section III-B)];
- a nonlinear cost function J_{NL}, minimizing the error between the amplitudes of the spatial directivity pattern $H(\omega, \theta)$ and the desired spatial directivity pattern $D(\omega, \theta)$, not taking into account the phase of the spatial directivity patterns [minimizing this cost function leads to a nonlinear optimization problem (cf. Section III-C)].

Obviously, it is also possible to use various other cost functions, which are, e.g., based on the "conventional" eigenfilter [19], [25], a maximum energy array [24], or a (nonlinear) minimax criterion [21]–[23].

We will consider the design of broadband beamformers over the total frequency-angle plane of interest, i.e., we will not split up the fullband problem into separate smallband problems for distinct frequencies. Furthermore, we will not approximate the double integrals that arise in the design by a finite sum over a grid of frequencies and angles, as, e.g., has been done in [20] for the nonlinear cost function J_{NL}. In [19], the three considered cost functions have been discussed in more detail for omni-directional microphones with a flat frequency response. Although the nonlinear cost function leads to the best performance, the computational complexity for computing the filter coefficients can be quite large, since an iterative optimization procedure is required. In [19], it has been shown that the TLS eigenfilter design procedure is the preferred noniterative design procedure, since it leads to a better performance than the weighted least-squares, the "conventional" eigenfilter and the maximum energy array design procedures.

For all considered cost functions, we will first discuss the *general design procedure* for an arbitrary desired spatial directivity pattern $D(\omega, \theta)$ and for frequency- and angle-dependent microphone characteristics $A_n(\omega, \theta)$. Next, the microphone characteristics will be assumed to be independent of frequency and angle, i.e., omni-directional, frequency-flat microphones. Even if this assumption is not exactly satisfied in practice, it is generally possible to split up the complete considered frequency-angle region into several smaller frequency-angle regions where this assumption holds. We will then focus on the *specific design case* of a broadband beamformer having a desired response $D(\omega, \theta) = 0$ in the stopband region (Ω_s, Θ_s) and $D(\omega, \theta) = 1$ in the passband region (Ω_p, Θ_p). For the specific design case, we will consider a weighting function $F(\omega, \theta) = 1$ in the passband and $F(\omega, \theta) = \alpha$ in the stopband.

A. Weighted Least-Squares (LS) Cost Function

1) General Design Procedure: Considering the least-squares (LS) error $|H(\omega, \theta) - D(\omega, \theta)|^2$, the weighted LS cost function (e.g., used in [32] for the design of FIR filters) is defined as

$$\boxed{J_{LS}(\mathbf{w}) = \int_\Theta \int_\Omega F(\omega, \theta) |H(\omega, \theta) - D(\omega, \theta)|^2 \, d\omega \, d\theta} \tag{19}$$

where both the phase and the amplitude of $H(\omega, \theta)$ are taken into account. $F(\omega, \theta)$ is a positive real weighting function, assigning more or less importance to certain frequencies or angles. This cost function can be written as

$$\begin{aligned} J_{LS}(\mathbf{w}) = &\int_\Theta \int_\Omega F(\omega, \theta) |H(\omega, \theta)|^2 \, d\omega \, d\theta \\ &+ \int_\Theta \int_\Omega F(\omega, \theta) |D(\omega, \theta)|^2 \, d\omega \, d\theta \\ &- 2 \int_\Theta \int_\Omega F(\omega, \theta) \, \mathrm{Re}\{D(\omega, \theta) H^*(\omega, \theta)\}. \end{aligned} \tag{20}$$

Using (17) and the fact that

$$\begin{aligned} &\mathrm{Re}\{D(\omega, \theta) H^*(\omega, \theta)\} \\ &\quad = \mathbf{w}^T [D_R(\omega, \theta) \overline{\mathbf{g}}_R(\omega, \theta) + D_I(\omega, \theta) \overline{\mathbf{g}}_I(\omega, \theta)] \end{aligned} \tag{21}$$

this cost function can be rewritten as the quadratic function

$$\boxed{J_{LS}(\mathbf{w}) = \mathbf{w}^T \overline{\mathbf{Q}}_{LS} \mathbf{w} - 2\mathbf{w}^T \overline{\mathbf{a}} + d_{LS}} \tag{22}$$

where

$$\overline{\mathbf{Q}}_{LS} = \int_\Theta \int_\Omega F(\omega, \theta) \overline{\mathbf{G}}_R(\omega, \theta) \, d\omega \, d\theta \tag{23}$$

$$\begin{aligned} \overline{\mathbf{a}} = \int_\Theta \int_\Omega F(\omega, \theta) [&D_R(\omega, \theta) \overline{\mathbf{g}}_R(\omega, \theta) \\ &+ D_I(\omega, \theta) \overline{\mathbf{g}}_I(\omega, \theta)] \, d\omega \, d\theta \end{aligned} \tag{24}$$

$$d_{LS} = \int_\Theta \int_\Omega F(\omega, \theta) |D(\omega, \theta)|^2 \, d\omega \, d\theta. \tag{25}$$

The filter $\mathbf{w}_{LS}$, minimizing the weighted LS cost function $J_{LS}(\mathbf{w})$, is given by

$$\boxed{\mathbf{w}_{LS} = \overline{\mathbf{Q}}_{LS}^{-1} \overline{\mathbf{a}}}. \tag{26}$$

2) Omni-Directional, Frequency-Flat Microphones: When the microphone characteristics are independent of frequency and angle, the diagonal matrices containing the microphone characteristics are $\mathbf{A}_R(\omega, \theta) = \mathbf{A}_R$ and $\mathbf{A}_I(\omega, \theta) = \mathbf{A}_I$. Using (12) and (18), the vector $\overline{\mathbf{a}}$ and the matrix $\overline{\mathbf{Q}}_{LS}$ are now equal to [assuming $D(\omega, \theta)$ to be real-valued]

$$\boxed{\begin{aligned} \overline{\mathbf{a}} &= \mathbf{A}_R \mathbf{a} - \mathbf{A}_I \mathbf{a}^\circ \\ \overline{\mathbf{Q}}_{LS} &= \mathbf{A}_R \mathbf{Q}_{LS} \mathbf{A}_R + \mathbf{A}_I \mathbf{Q}_{LS} \mathbf{A}_I - \mathbf{A}_I \mathbf{Q}_{LS}^\circ \mathbf{A}_R \\ &\quad + \mathbf{A}_R \mathbf{Q}_{LS}^\circ \mathbf{A}_I \end{aligned}} \tag{27}$$

where

$$\mathbf{a} = \int_\Theta \int_\Omega F(\omega, \theta) D(\omega, \theta) \mathbf{g}_R(\omega, \theta) \, d\omega \, d\theta \quad (\ldots)$$

$$\mathbf{Q}_{LS} = \int_\Theta \int_\Omega F(\omega, \theta) \mathbf{G}_R(\omega, \theta) \, d\omega \, d\theta \tag{28}$$

$$\mathbf{a}^\circ = \int_\Theta \int_\Omega F(\omega,\theta) D(\omega,\theta) \mathbf{g}_I(\omega,\theta)\, d\omega\, d\theta \quad (\ldots)$$

$$\mathbf{Q}_{LS}^\circ = \int_\Theta \int_\Omega F(\omega,\theta) \mathbf{G}_I(\omega,\theta)\, d\omega\, d\theta. \quad (29)$$

The ith element of $\overline{\mathbf{a}}$ and the (i, j)th element of $\overline{\mathbf{Q}}_{LS}$ are equal to

$$\overline{\mathbf{a}}^i = a_n \left(\cos\gamma_n \mathbf{a}^i + \sin\gamma_n \mathbf{a}^{\circ, i}\right) \quad (30)$$

$$\overline{\mathbf{Q}}_{LS}^{ij} = a_n a_m \Big(\cos(\gamma_n - \gamma_m)\mathbf{Q}_{LS}^{ij} + \sin(\gamma_n - \gamma_m)\mathbf{Q}_{LS}^{\circ, ij}\Big) \quad (31)$$

where $n = \lfloor (i-1)/L \rfloor$, and $m = \lfloor (j-1)/L \rfloor$.

3) Specific Design Case: For the specific design case where $D(\omega,\theta) = 1$, $F(\omega,\theta) = 1$ in the passband and $D(\omega,\theta) = 0$, $F(\omega,\theta) = \alpha$ in the stopband, (28)–(31) become

$$\mathbf{a} = \int_{\Theta_p} \int_{\Omega_p} \mathbf{g}_R(\omega,\theta)\, d\omega\, d\theta \quad (\ldots)$$

$$\mathbf{a}^\circ = \int_{\Theta_p} \int_{\Omega_p} \mathbf{g}_I(\omega,\theta)\, d\omega\, d\theta \quad (32)$$

$$\mathbf{Q}_{LS} = \underbrace{\int_{\Theta_p} \int_{\Omega_p} \mathbf{G}_R(\omega,\theta)\, d\omega\, d\theta}_{\mathbf{Q}_e^p} + \alpha \underbrace{\int_{\Theta_s} \int_{\Omega_s} \mathbf{G}_R(\omega,\theta)\, d\omega\, d\theta}_{\mathbf{Q}_e^s} \quad (33)$$

$$\mathbf{Q}_{LS}^\circ = \underbrace{\int_{\Theta_p} \int_{\Omega_p} \mathbf{G}_I(\omega,\theta)\, d\omega\, d\theta}_{\mathbf{Q}_e^{\circ,p}} + \alpha \underbrace{\int_{\Theta_s} \int_{\Omega_s} \mathbf{G}_I(\omega,\theta)\, d\omega\, d\theta}_{\mathbf{Q}_e^{\circ,s}}. \quad (34)$$

The ith element of $\mathbf{a}$ and the (i, j)th element of $\mathbf{Q}_e$, i.e., $\mathbf{Q}_e^p$ or $\mathbf{Q}_e^s$, can be computed as

$$\mathbf{a}^i = \int_{\Theta_p} \int_{\Omega_p} \mathbf{g}_R^i(\omega,\theta)\, d\omega\, d\theta = \int_{\Theta_p} \int_{\Omega_p} \cos\left[\omega\left(k + \frac{d_n \cos\theta}{c} f_s\right)\right] d\omega\, d\theta \quad (35)$$

$$\mathbf{Q}_e^{ij} = \int_\Theta \int_\Omega \mathbf{G}_R^{ij}(\omega,\theta)\, d\omega\, d\theta = \int_\Theta \int_\Omega \cos\left[\omega\left((k-l) + \frac{(d_n - d_m)\cos\theta}{c} f_s\right)\right] d\omega\, d\theta \quad (36)$$

where $k = \text{mod}(i-1, L)$, $l = \text{mod}(j-1, L)$, $n = \lfloor (i-1)/L \rfloor$, and $m = \lfloor (j-1)/L \rfloor$. Similarly, the ith element of $\mathbf{a}^\circ$ and the (i, j)th element of $\mathbf{Q}_e^\circ$ can be computed by replacing $\cos(\cdot)$ with $-\sin(\cdot)$ in the integrals of (35) and (36). All these integrals can be considered to be special cases of the integral

$$\int_{\theta_1}^{\theta_2} \int_{\omega_1}^{\omega_2} \cos[\omega(\alpha + \beta\cos\theta) + \gamma]\, d\omega\, d\theta \quad (37)$$

of which the computation is discussed in Appendix A.

B. TLS Eigenfilter Cost Function

Eigenfilters have been introduced for designing 1-D linear-phase FIR filters [33] and 2-D FIR filters [34], [35]. In [19] and [25], two noniterative broadband beamformer design procedures based on eigenfilters have been discussed. The "conventional" eigenfilter technique minimizes the error between the spatial directivity patterns $H(\omega,\theta)$ and $D(\omega,\theta)H(\omega_c,\theta_c)/D(\omega_c,\theta_c)$ and requires a reference frequency-angle point (ω_c, θ_c). The TLS eigenfilter minimizes the total least-squares (TLS) error between the spatial directivity pattern $H(\omega,\theta)$ and the desired spatial directivity pattern $D(\omega,\theta)$ and does not require a reference point. In [19], it has been shown that the performance of the TLS eigenfilter always exceeds the performance of the weighted LS and the "conventional" eigenfilter cost functions and therefore is the preferred noniterative design procedure.

The TLS eigenfilter cost function is defined as

$$\boxed{J_{\text{TLS}}(\mathbf{w}) = \frac{\int_\Theta \int_\Omega F(\omega,\theta) |D(\omega,\theta) - H(\omega,\theta)|^2\, d\omega\, d\theta}{\int_\Theta \int_\Omega |H(\omega,\theta)|^2\, d\omega\, d\theta + 1}} \quad (38)$$

which can be written as

$$J_{\text{TLS}}(\mathbf{w}) = \int_\Theta \int_\Omega F(\omega,\theta) \frac{|D(\omega,\theta) - H(\omega,\theta)|^2}{\mathbf{w}^T \overline{\mathbf{Q}}_e^{tot} \mathbf{w} + 1}\, d\omega\, d\theta \quad (39)$$

where $\overline{\mathbf{Q}}_e^{tot}$ is defined as

$$\overline{\mathbf{Q}}_e^{tot} = \int_\Theta \int_\Omega \overline{\mathbf{G}}_R(\omega,\theta)\, d\omega\, d\theta. \quad (40)$$

For computing the TLS error, the expression $\mathbf{w}^T \overline{\mathbf{Q}}_e^{tot} \mathbf{w}$ is used in the denominator of (39) instead of the usual expression $\mathbf{w}^T\mathbf{w}$ since $\mathbf{w}^T \overline{\mathbf{Q}}_e^{tot} \mathbf{w}$ represents the total area under the spatial directivity spectrum $|H(\omega,\theta)|^2$. The TLS eigenfilter cost function in (39) can be rewritten as the Rayleigh-quotient

$$\boxed{J_{\text{TLS}}(\mathbf{w}) = \frac{\hat{\mathbf{w}}^T \hat{\mathbf{Q}}_{\text{TLS}} \hat{\mathbf{w}}}{\hat{\mathbf{w}}^T \hat{\mathbf{Q}}_e^{tot} \hat{\mathbf{w}}}} \quad (41)$$

where

$$\hat{\mathbf{w}} = \begin{bmatrix} \mathbf{w} \\ -1 \end{bmatrix}, \qquad \hat{\mathbf{Q}}_{\text{TLS}} = \begin{bmatrix} \overline{\mathbf{Q}}_{LS} & \overline{\mathbf{a}} \\ \overline{\mathbf{a}}^T & d_{LS} \end{bmatrix}$$
$$\hat{\mathbf{Q}}_e^{tot} = \begin{bmatrix} \overline{\mathbf{Q}}_e^{tot} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \quad (42)$$

with $\overline{\mathbf{Q}}_{LS}$, $\overline{\mathbf{a}}$, and d_{LS} defined in Section III-A. The filter $\hat{\mathbf{w}}_{\text{TLS}}$ minimizing $J_{\text{TLS}}(\mathbf{w})$ in (41) is the generalized eigenvector of $\hat{\mathbf{Q}}_{\text{TLS}}$ and $\hat{\mathbf{Q}}_e^{tot}$, corresponding to the smallest generalized eigenvalue. After scaling the last element of $\hat{\mathbf{w}}_{\text{TLS}}$ to -1, the actual solution $\mathbf{w}_{\text{TLS}}$ is obtained as the first M elements of $\hat{\mathbf{w}}_{\text{TLS}}$.

In the case of omni-directional, frequency-flat microphones, and for the specific design case, we can use similar expressions as derived in Sections III-A2 and 3.

C. Nonlinear Criterion

1) General Design Procedure: Instead of minimizing the LS error $|H(\omega, \theta) - D(\omega, \theta)|^2$ or the TLS error, one can also minimize the error between the amplitudes $|H(\omega, \theta)| - |D(\omega, \theta)|$ because in general, the phase of the spatial directivity pattern is not relevant. This problem formulation leads to the cost function

$$\tilde{J}_{\mathrm{NL}}(\mathbf{w}) = \int_\Theta \int_\Omega F(\omega, \theta)[|H(\omega, \theta)| - |D(\omega, \theta)|]^2 \, d\omega \, d\theta \tag{43}$$

and gives rise to a nonlinear optimization problem, which has to be solved using iterative optimization techniques. These iterative optimization techniques generally involve several evaluations of $\tilde{J}_{\mathrm{NL}}(\mathbf{w})$ in each iteration step. A complexity problem now arises because the filter coefficients $\mathbf{w}$ can not be extracted from the double integral because of the square root in the term $-2F(\omega, \theta)|D(\omega, \theta)|\,|H(\omega, \theta)| = -2F(\omega, \theta)|D(\omega, \theta)|\sqrt{\mathbf{w}^T\overline{\mathbf{G}}_R(\omega, \theta)\mathbf{w}}$ [19]. Hence, for every intermediate $\mathbf{w}$, the double integrals have to be recomputed numerically, which is a computationally very demanding procedure. However, by slightly changing the nonlinear cost function in (43), it is possible to overcome this computational problem: Instead of minimizing the error between the amplitudes $|H(\omega, \theta)|$ and $|D(\omega, \theta)|$, it is also feasible to minimize the error between the square of the amplitudes $|H(\omega, \theta)|^2$ and $|D(\omega, \theta)|^2$ and define the cost function

$$\boxed{J_{\mathrm{NL}}(\mathbf{w}) = \int_\Theta \int_\Omega F(\omega,\theta)[|H(\omega,\theta)|^2 - |D(\omega,\theta)|^2]^2 \, d\omega \, d\theta} \tag{44}$$

which is again independent of the phase of the spatial directivity patterns. Using (13) and (17), the cost function $J_{\mathrm{NL}}(\mathbf{w})$ can be written as

$$\begin{aligned} J_{\mathrm{NL}}(\mathbf{w}) &= \int_\Theta \int_\Omega F(\omega, \theta)\left(\mathbf{w}^T\overline{\mathbf{G}}(\omega, \theta)\mathbf{w}\right)^2 d\omega \, d\theta \\ &\quad + \int_\Theta \int_\Omega F(\omega, \theta)|D(\omega, \theta)|^4 \, d\omega \, d\theta \\ &\quad - 2\mathbf{w}^T \int_\Theta \int_\Omega F(\omega,\theta)|D(\omega,\theta)|^2\overline{\mathbf{G}}_R(\omega,\theta)\, d\omega \, d\theta \mathbf{w} \\ &= \overline{J}_{sum}(\mathbf{w}) + d_{\mathrm{NL}} - 2\mathbf{w}^T\overline{\mathbf{Q}}_{\mathrm{NL}}\mathbf{w} \end{aligned} \tag{46}$$

with

$$\begin{aligned} \overline{J}_{sum}(\mathbf{w}) &= \int_\Theta \int_\Omega F(\omega, \theta)|H(\omega, \theta)|^4 \, d\omega \, d\theta \\ &= \int_\Theta \int_\Omega F(\omega, \theta)\left(\mathbf{w}^T\overline{\mathbf{G}}(\omega, \theta)\mathbf{w}\right)^2 d\omega \, d\theta \end{aligned} \tag{47}$$

$$d_{\mathrm{NL}} = \int_\Theta \int_\Omega F(\omega, \theta)|D(\omega, \theta)|^4 \, d\omega \, d\theta \tag{48}$$

$$\overline{\mathbf{Q}}_{\mathrm{NL}} = \int_\Theta \int_\Omega F(\omega, \theta)|D(\omega, \theta)|^2\overline{\mathbf{G}}_R(\omega, \theta)\, d\omega \, d\theta. \tag{49}$$

The cost function $J_{\mathrm{NL}}(\mathbf{w})$ can be minimized using iterative optimization techniques, which are discussed in Section III-C3. As will be shown in Section III-C2, the filter coefficients $\mathbf{w}$ can be extracted from the double integral in (47), such that these double integrals only need to be computed once.

2) Omni-Directional, Frequency-Flat Microphones: When the microphone characteristics are independent of frequency and angle, the matrix $\overline{\mathbf{Q}}_{\mathrm{NL}}$ can be computed similarly as $\overline{\mathbf{Q}}_{LS}$ in (27) as

$$\begin{aligned} \overline{\mathbf{Q}}_{\mathrm{NL}} = \mathbf{A}_R\mathbf{Q}_{\mathrm{NL}}\mathbf{A}_R + \mathbf{A}_I\mathbf{Q}_{\mathrm{NL}}\mathbf{A}_I \\ -\mathbf{A}_I\mathbf{Q}_{\mathrm{NL}}^\circ\mathbf{A}_R + \mathbf{A}_R\mathbf{Q}_{\mathrm{NL}}^\circ\mathbf{A}_I \end{aligned} \tag{50}$$

where

$$\mathbf{Q}_{\mathrm{NL}} = \int_\Theta \int_\Omega F(\omega, \theta)|D(\omega, \theta)|^2\mathbf{G}_R(\omega, \theta)\, d\omega \, d\theta \tag{51}$$

$$\mathbf{Q}_{\mathrm{NL}}^\circ = \int_\Theta \int_\Omega F(\omega, \theta)|D(\omega, \theta)|^2\mathbf{G}_I(\omega, \theta)\, d\omega \, d\theta. \tag{52}$$

Using (16), the expression $|H(\omega, \theta)|^4$, arising in the computation of $\overline{J}_{sum}(\mathbf{w})$ can be written as

$$|H(\omega, \theta)|^4 = \left(\mathbf{w}^T\overline{\mathbf{G}}(\omega, \theta)\mathbf{w}\right)\left(\mathbf{w}^T\overline{\mathbf{G}}(\omega, \theta)\mathbf{w}\right) \tag{53}$$

$$\begin{aligned} &= \left(\sum_{i=1}^M \sum_{j=1}^M w(i)w(j)\overline{\mathbf{G}}^{ij}(\omega, \theta)\right) \\ &\quad \cdot \left(\sum_{k=1}^M \sum_{l=1}^M w(k)w(l)\overline{\mathbf{G}}^{kl}(\omega, \theta)\right) \end{aligned} \tag{54}$$

$$\begin{aligned} &= \sum_{i=1}^M \sum_{j=1}^M \sum_{k=1}^M \sum_{l=1}^M w(i)w(j)w(k)w(l)a_{ijkl} \\ &\quad \cdot \exp(-j[\omega(\alpha_{ijkl} + \beta_{ijkl}\cos\theta) + \gamma_{ijkl}]) \end{aligned} \tag{55}$$

where

$$\boxed{\begin{aligned} \alpha_{ijkl} &= \mathrm{mod}(i-1, L) - \mathrm{mod}(j-1, L) \\ &\quad +\mathrm{mod}(k-1, L) - \mathrm{mod}(l-1, L) \\ \beta_{ijkl} &= \left(d_{\lfloor (i-1)/L \rfloor} - d_{\lfloor (j-1)/L \rfloor} + d_{\lfloor (k-1)/L \rfloor} \right. \\ &\quad \left. -d_{\lfloor (l-1)/L \rfloor}\right) f_s/c \\ a_{ijkl} &= a_{\lfloor (i-1)/L \rfloor} \cdot a_{\lfloor (j-1)/L \rfloor} \cdot a_{\lfloor (k-1)/L \rfloor} \cdot a_{\lfloor (l-1)/L \rfloor} \\ \gamma_{ijkl} &= \gamma_{\lfloor (i-1)/L \rfloor} - \gamma_{\lfloor (j-1)/L \rfloor} + \gamma_{\lfloor (k-1)/L \rfloor} \\ &\quad -\gamma_{\lfloor (l-1)/L \rfloor} \end{aligned}}. \tag{56}$$

Since $|H(\omega, \theta)|^4$ is real, only the real part of the exponential function in (55) has to be considered, i.e.,

$$\begin{aligned} |H(\omega, \theta)|^4 &= \sum_{i=1}^M \sum_{j=1}^M \sum_{k=1}^M \sum_{l=1}^M w(i)w(j)w(k)w(l)a_{ijkl} \\ &\quad \cdot (\cos[\omega(\alpha_{ijkl} + \beta_{ijkl}\cos\theta)] \cdot \cos\gamma_{ijkl} \\ &\quad - \sin[\omega(\alpha_{ijkl} + \beta_{ijkl}\cos\theta)] \cdot \sin\gamma_{ijkl}). \end{aligned} \tag{57}$$

Hence, $\overline{J}_{sum}(\mathbf{w})$ can be written as

$$\boxed{\begin{aligned} \overline{J}_{sum}(w) &= \sum_{i=1}^M \sum_{j=1}^M \sum_{k=1}^M \sum_{l=1}^M w(i)w(j)w(k)w(l) \\ &\quad \cdot \underbrace{a_{ijkl}(\cos\gamma_{ijkl} \cdot \rho_{ijkl} - \sin\gamma_{ijkl} \cdot \rho_{ijkl}^\circ)}_{\overline{\rho}_{ijkl}} \end{aligned}} \tag{58}$$

where

$$\rho_{ijkl} = \int_{\Theta}\int_{\Omega} F(\omega,\theta)\cos[\omega(\alpha_{ijkl}+\beta_{ijkl}\cos\theta)]\,d\omega\,d\theta \tag{59}$$

$$\rho^{\circ}_{ijkl} = \int_{\Theta}\int_{\Omega} F(\omega,\theta)\sin[\omega(\alpha_{ijkl}+\beta_{ijkl}\cos\theta)]\,d\omega\,d\theta. \tag{60}$$

The double integrals in (59) and (60) only need to be computed once (since ρ_{ijkl} and ρ°_{ijkl} are independent of $\mathbf{w}$). Therefore, the function $\overline{J}_{sum}(\mathbf{w})$, and, hence, also the total cost function $J_{\mathrm{NL}}(\mathbf{w})$, can be evaluated without having to calculate double integrals for every $\mathbf{w}$. This result also remains true when the microphone characteristics are frequency- and angle-dependent.

3) Nonlinear Optimization Technique: Minimizing $J_{\mathrm{NL}}(\mathbf{w})$ requires an iterative nonlinear optimization technique, for which we have used both a medium-scale quasi-Newton method with cubic polynomial line search and a large-scale subspace trust region method [36], [37]. In order to improve the numerical robustness and the convergence speed, both the gradient and the Hessian

$$\frac{\partial J_{\mathrm{NL}}(\mathbf{w})}{\partial \mathbf{w}} = \frac{\partial \overline{J}_{sum}(\mathbf{w})}{\partial \mathbf{w}} - 4\overline{\mathbf{Q}}_{\mathrm{NL}}\mathbf{w} \tag{61}$$

$$\mathbf{H}_{\mathrm{NL}}(\mathbf{w}) = \frac{\partial^2 J_{\mathrm{NL}}(\mathbf{w})}{\partial^2 \mathbf{w}} = \frac{\partial^2 \overline{J}_{sum}(\mathbf{w})}{\partial^2 \mathbf{w}} - 4\overline{\mathbf{Q}}_{\mathrm{NL}} \tag{62}$$

can be supplied analytically. In [18] and [19], it has been shown that $\partial\overline{J}_{sum}(\mathbf{w})/\partial\mathbf{w}$ can be calculated as

$$\frac{\partial \overline{J}_{sum}(\mathbf{w})}{\partial \mathbf{w}} = 4\overline{\mathbf{Q}}_{sum}(\mathbf{w})\cdot\mathbf{w} \tag{63}$$

with the $(m,\,n)$th element of $\overline{\mathbf{Q}}_{sum}(\mathbf{w})$ equal to

$$\overline{\mathbf{Q}}^{mn}_{sum}(\mathbf{w}) = \sum_{i=1}^{M}\sum_{j=1}^{M} w(i)w(j)\overline{\rho}_{ijmn} \tag{64}$$

and the $(m,\,n)$th element of $\partial^2\overline{J}_{sum}(\mathbf{w})/\partial^2\mathbf{w}$ equal to

$$\frac{\partial^2 \overline{J}_{sum}(\mathbf{w})}{\partial w(m)\partial w(n)} = 4\sum_{i=1}^{M}\sum_{j=1}^{M} w(i)w(j)\left(2\overline{\rho}_{ijmn}+\overline{\rho}_{imjn}\right). \tag{65}$$

Hence, stationary points $\mathbf{w}_s$, i.e., filter coefficients $\mathbf{w}$ for which the gradient is 0, satisfy

$$\left(\overline{\mathbf{Q}}_{sum}(\mathbf{w}_s) - \overline{\mathbf{Q}}_{\mathrm{NL}}\right)\mathbf{w}_s = \mathbf{0}. \tag{66}$$

In addition, it can be shown that the quadratic form $\mathbf{w}^T\mathbf{H}_{\mathrm{NL}}(\mathbf{w})\mathbf{w}$ in a stationary point $\mathbf{w}_s$ is equal to

$$\begin{aligned}\mathbf{w}_s^T\mathbf{H}_{\mathrm{NL}}(\mathbf{w}_s)\mathbf{w}_s &= 12\mathbf{w}_s^T\overline{\mathbf{Q}}_{sum}(\mathbf{w}_s)\mathbf{w}_s - 4\mathbf{w}_s^T\overline{\mathbf{Q}}_{\mathrm{NL}}\mathbf{w}_s\\ &= 8\mathbf{w}_s^T\overline{\mathbf{Q}}_{\mathrm{NL}}\mathbf{w}_s.\end{aligned} \tag{67}$$

Since, in general, the matrix $\overline{\mathbf{Q}}_{\mathrm{NL}}$, defined in (49), is positive-definite, the quadratic form $\mathbf{w}_s^T\mathbf{H}_{\mathrm{NL}}(\mathbf{w}_s)\mathbf{w}_s$ is strictly positive in all stationary points, except for $\mathbf{w}_s = \mathbf{0}$, where it is equal to zero. Therefore, *all stationary points are either local minima or saddle points* (except for $\mathbf{w}_s = \mathbf{0}$, where the Hessian is negative-definite, such that it is the only local maximum). Simulations have indicated that for each design problem, a number of local minima exist, which are generally related to the symmetry present in the considered problem. However, the cost function in all local minima seems to be approximately equal, such that any of these local minima can be used as the final solution for the broadband beamformer.

4) Specific Design Case: For the specific design case considered in Section III-A3, the matrices $\mathbf{Q}_{\mathrm{NL}}$ and $\mathbf{Q}^{\circ}_{\mathrm{NL}}$ and the scalars d_{NL}, ρ_{ijkl} and ρ°_{ijkl} are equal to

$$\mathbf{Q}_{\mathrm{NL}} = \mathbf{Q}^p_e, \qquad \mathbf{Q}^{\circ}_{\mathrm{NL}} = \mathbf{Q}^{\circ,p}_e, \qquad d_{\mathrm{NL}} = d_{LS} \tag{68}$$

$$\begin{aligned}\rho_{ijkl} = &\int_{\Theta_p}\int_{\Omega_p}\cos[\omega(\alpha_{ijkl}+\beta_{ijkl}\cos\theta)]\,d\omega\,d\theta\\ &+\alpha\int_{\Theta_s}\int_{\Omega_s}\cos[\omega(\alpha_{ijkl}+\beta_{ijkl}\cos\theta)]\,d\omega\,d\theta\end{aligned} \tag{69}$$

$$\begin{aligned}\rho^{\circ}_{ijkl} = &\int_{\Theta_p}\int_{\Omega_p}\sin[\omega(\alpha_{ijkl}+\beta_{ijkl}\cos\theta)]\,d\omega\,d\theta\\ &+\alpha\int_{\Theta_s}\int_{\Omega_s}\sin[\omega(\alpha_{ijkl}+\beta_{ijkl}\cos\theta)]\,d\omega\,d\theta\end{aligned} \tag{70}$$

where $\mathbf{Q}^p_e$, $\mathbf{Q}^{\circ,p}_e$, and d_{LS} have been defined in Section III-A. The integrals in (69) and (70) can be considered to be special cases of the integral (37), of which the computation is discussed in Appendix A.

IV. Robust Broadband Beamforming

Using the cost functions presented in Section III, it is possible to design broadband beamformers with an arbitrary spatial directivity pattern $D(\omega,\,\theta)$, when the microphone characteristics $A_n(\omega,\,\theta)$ are exactly known (and fixed). However, these beamformers are known to be highly sensitive to errors in the microphone array characteristics (gain, phase, and microphone position) [15], [26], [29]. Small deviations from the assumed microphone array characteristics can lead to large deviations from the desired spatial directivity pattern, especially when using small-size microphone arrays, e.g., in hearing aids and cochlear implants (cf. Section V). Since, in practice, it is difficult to manufacture microphones having exactly the same characteristics, it is practically impossible to exactly know the microphone array characteristics without a measurement or a calibration procedure. Obviously, the cost of such a calibration procedure for every individual microphone array is objectionable. Moreover, after calibration, the microphone characteristics can still drift over time [26].

Instead of measuring or calibrating every individual microphone array, it is better to consider all feasible microphone characteristics (in this paper, we only consider gain and phase[2]) and to either optimize one of the following criteria.

- The *mean performance*, i.e., the weighted sum of the cost functions for all feasible microphone characteristics, using the probability of the microphone characteristics as weights (cf. Section IV-A). This procedure requires knowledge of the gain and the phase probability density functions (pdf), which can, e.g., be obtained from the microphone manufacturers. It will be shown that for gain errors only the moments of the gain pdf are required,

[2]Robustness against microphone position errors can actually be considered a special case of robustness against phase errors since a position error corresponds to a frequency- and angle-dependent phase error [38].

whereas in general, for phase errors complete knowledge of the phase pdf is required. We will apply this mean performance criterion to the weighted LS and the nonlinear cost function, whereas it is not straightforward to apply this criterion to the TLS eigenfilter cost function. When optimizing this mean performance criterion, it is, however, still possible that for some specific gain/phase combination (typically with a low probability), the cost function is quite high.

- The *worst-case performance*, i.e., the maximum cost function for all feasible microphone characteristics, leading to a minimax criterion (cf. Section IV-B). This is a stronger criterion, since the cost for the worst-case scenario is minimized. We will apply this criterion to all considered cost functions.

The same problem of gain and phase errors has been studied in [27]. However, in [27], only the narrowband case for a specific directivity pattern, with a uniform pdf and a LS cost function, has been considered. The approach presented here is more general in the sense that we consider broadband beamformers with an arbitrary spatial directivity pattern, arbitrary probability density functions, and several cost functions.

A. Weighted Sum Using Probability Density Functions

The total cost function $J_{\text{tot}}(\mathbf{w})$ is defined as the weighted sum of the cost functions for all feasible microphone characteristics, using the probability of the microphone characteristics as weights, i.e.,

$$\boxed{\begin{aligned} J_{\text{tot}}(\mathbf{w}) = \int_{A_0} \cdots \int_{A_{N-1}} & J(\mathbf{w}, A_0, \ldots, A_{N-1}) \\ & \cdot f_{\mathcal{A}}(A_0) \cdots f_{\mathcal{A}}(A_{N-1})\, dA_0 \cdots dA_{N-1} \end{aligned}} \tag{71}$$

where $J(\mathbf{w}, A_0, \ldots, A_{N-1})$ is the cost function for a specific microphone characteristic $\{A_0, \ldots, A_{N-1}\}$, and $f_{\mathcal{A}}(A)$ is the probability density function (pdf) of the stochastic variable $A = ae^{-j\gamma}$, i.e., the joint pdf of the stochastic variables a (gain) and γ (phase), $f_{\mathcal{A}}(A) = f_{\alpha,\mathcal{G}}(a, \gamma)$. We assume that $f_{\mathcal{A}}(A)$ is independent of frequency and angle or that $f_{\mathcal{A}}(A)$ is available for different frequency-angle regions, such that the problem can easily be split up. Without loss of generality, we also assume that all microphone characteristics $A_n, n = 0 \cdots N-1$, are described by the same pdf $f_{\mathcal{A}}(A)$. Furthermore, we assume that a and γ are independent stochastic variables such that the joint pdf is separable, i.e., $f_{\mathcal{A}}(A) = f_\alpha(a) f_{\mathcal{G}}(\gamma)$, where $f_\alpha(a)$ is the pdf of the gain a, and $f_{\mathcal{G}}(\gamma)$ is the pdf of the phase γ. For these pdfs, the relation

$$\int_a f_\alpha(a)\, da = 1, \qquad \int_\gamma f_{\mathcal{G}}(\gamma)\, d\gamma = 1 \tag{72}$$

holds. We will consider two cost functions from Section III: the weighted LS and the nonlinear cost function (it is not straightforward to apply this criterion to the TLS eigenfilter cost function). Remarkably, the same design procedures as for the nonrobust design in Section III can be used, and we only require some additional parameters, which can be easily calculated from the gain and the phase pdf.

1) Weighted LS Cost Function: The mean performance weighted LS cost function can be written as

$$\begin{aligned} J_{LS}^{tot}(\mathbf{w}) = \int_{A_0} \cdots \int_{A_{N-1}} & J_{LS}(\mathbf{w}, A_0, \ldots, A_{N-1}) \\ & \cdot f_{\mathcal{A}}(A_0) \cdots f_{\mathcal{A}}(A_{N-1})\, dA_0 \cdots dA_{N-1}. \end{aligned} \tag{73}$$

The cost function $J_{LS}(\mathbf{w}, A_0, \ldots, A_{N-1})$ for a specific microphone characteristic is equal to (22), i.e.,

$$J_{LS}(\mathbf{w}, A_0, \ldots, A_{N-1}) = \mathbf{w}^T \overline{\mathbf{Q}}_{LS} \mathbf{w} - 2\mathbf{w}^T \overline{\mathbf{a}} + d_{LS}. \tag{74}$$

By combining (73) and (74), the mean performance weighted LS cost function can be written as

$$\begin{aligned} J_{LS}^{tot}(\mathbf{w}) = \mathbf{w}^T & \int_{A_0} \cdots \int_{A_{N-1}} \overline{\mathbf{Q}}_{LS} f_{\mathcal{A}}(A_0) \cdots \\ & f_{\mathcal{A}}(A_{N-1})\, dA_0 \cdots dA_{N-1} \mathbf{w} \\ & - 2\mathbf{w}^T \int_{A_0} \cdots \int_{A_{N-1}} \overline{\mathbf{a}} f_{\mathcal{A}}(A_0) \cdots \\ & f_{\mathcal{A}}(A_{N-1})\, dA_0 \cdots dA_{N-1} \\ & + \int_{A_0} \cdots \int_{A_{N-1}} d_{LS} f_{\mathcal{A}}(A_0) \cdots \\ & f_{\mathcal{A}}(A_{N-1})\, dA_0 \cdots dA_{N-1} \end{aligned} \tag{75}$$

$$= \mathbf{w}^T \overline{\mathbf{Q}}_{\text{tot}} \mathbf{w} - 2\mathbf{w}^T \overline{\mathbf{a}}_{\text{tot}} + d_{LS} \tag{76}$$

which has the same form as (22). Using (30), the ith element of $\overline{\mathbf{a}}_{\text{tot}}$ is equal to

$$\begin{aligned} \overline{\mathbf{a}}_{tot}^i = \int_{A_0} \cdots \int_{A_{N-1}} & a_n(\cos\gamma_n \mathbf{a}^i + \sin\gamma_n \mathbf{a}^{\circ,i}) \\ & \cdot f_{\mathcal{A}}(A_0) \cdots f_{\mathcal{A}}(A_{N-1})\, dA_0 \cdots dA_{N-1} \end{aligned} \tag{77}$$

$$= \int_{A_n} a_n(\cos\gamma_n \mathbf{a}^i + \sin\gamma_n \mathbf{a}^{\circ,i}) f_{\mathcal{A}}(A_n)\, dA_n \tag{78}$$

$$\begin{aligned} = \int_{a_n} a_n f_\alpha(a_n) da_n \Bigg[& \int_{\gamma_n} \cos\gamma_n f_{\mathcal{G}}(\gamma_n)\, d\gamma_n \mathbf{a}^i \\ & + \int_{\gamma_n} \sin\gamma_n f_{\mathcal{G}}(\gamma_n)\, d\gamma_n \mathbf{a}^{\circ,i} \Bigg] \end{aligned} \tag{79}$$

$$= \mu_a \left[\mu_\gamma^c \mathbf{a}^i + \mu_\gamma^s \mathbf{a}^{\circ,i}\right] \tag{80}$$

where

$$\boxed{\begin{aligned} & \mu_a = \int_a a f_\alpha(a)\, da, \qquad \mu_\gamma^c = \int_\gamma \cos\gamma f_{\mathcal{G}}(\gamma)\, d\gamma, \\ & \mu_\gamma^s = \int_\gamma \sin\gamma f_{\mathcal{G}}(\gamma)\, d\gamma \end{aligned}} \tag{81}$$

such that

$$\boxed{\overline{\mathbf{a}}_{\text{tot}} = \mu_a \mu_\gamma^c \mathbf{a} + \mu_a \mu_\gamma^s \mathbf{a}^\circ}. \tag{82}$$

Using (31), the (i, j)th element of $\overline{\mathbf{Q}}_{\text{tot}}$ is equal to

$$\begin{aligned} \overline{\mathbf{Q}}_{tot}^{ij} = \int_{A_0} \cdots \int_{A_{N-1}} & a_n a_m \Big(\cos(\gamma_n - \gamma_m) \mathbf{Q}_{LS}^{ij} \\ & + \sin(\gamma_n - \gamma_m) \mathbf{Q}_{LS}^{\circ,ij}\Big) \\ & \cdot f_{\mathcal{A}}(A_0) \cdots f_{\mathcal{A}}(A_{N-1})\, dA_0 \cdots dA_{N-1} \end{aligned} \tag{83}$$

$$= \int_{a_n}\int_{a_m} a_n a_m f_\alpha(a_n) f_\alpha(a_m)\, da_n\, da_m \cdot \left[\int_{\gamma_n}\int_{\gamma_m} \cos(\gamma_n-\gamma_m) f_{\mathcal{G}}(\gamma_n) f_{\mathcal{G}}(\gamma_m) \cdot d\gamma_n d\gamma_m \mathbf{Q}_{LS}^{ij} + \int_{\gamma_n}\int_{\gamma_m} \sin(\gamma_n-\gamma_m) \cdot f_{\mathcal{G}}(\gamma_n) f_{\mathcal{G}}(\gamma_m)\, d\gamma_n\, d\gamma_m \mathbf{Q}_{LS}^{\circ,ij}\right]. \quad (84)$$

If $n = m$, $\overline{\mathbf{Q}}_{tot}^{ij}$ is equal to

$$\overline{\mathbf{Q}}_{tot}^{ij} = \int_{a_n} a_n^2 f_\alpha(a_n)\, da_n \mathbf{Q}_{LS}^{ij} = \sigma_a^2 \mathbf{Q}_{LS}^{ij} \quad (85)$$

where σ_a^2 is the variance of the gain pdf, i.e.,

$$\boxed{\sigma_a^2 = \int_a a^2 f_\alpha(a)\, da} \quad (86)$$

whereas, if $n \neq m$, $\overline{\mathbf{Q}}_{tot}^{ij}$ is equal to

$$\overline{\mathbf{Q}}_{tot}^{ij} = \mu_a^2 \left[\sigma_\gamma^c \mathbf{Q}_{LS}^{ij} + \sigma_\gamma^s \mathbf{Q}_{LS}^{\circ,ij}\right] \quad (87)$$

where μ_a is the mean of the gain pdf, and

$$\sigma_\gamma^c = \int_{\gamma_1}\int_{\gamma_2} \cos(\gamma_1-\gamma_2) f_{\mathcal{G}}(\gamma_1) f_{\mathcal{G}}(\gamma_2)\, d\gamma_1\, d\gamma_2 \quad (88)$$

$$= \int_{\gamma_1}\int_{\gamma_2} (\cos\gamma_1 \cos\gamma_2 + \sin\gamma_1 \sin\gamma_2) \cdot f_{\mathcal{G}}(\gamma_1) f_{\mathcal{G}}(\gamma_2)\, d\gamma_1\, d\gamma_2 \quad (89)$$

$$\sigma_\gamma^s = \int_{\gamma_1}\int_{\gamma_2} \sin(\gamma_1-\gamma_2) f_{\mathcal{G}}(\gamma_1) f_{\mathcal{G}}(\gamma_2)\, d\gamma_1\, d\gamma_2 \quad (90)$$

$$= \int_{\gamma_1}\int_{\gamma_2} (\sin\gamma_1 \cos\gamma_2 - \cos\gamma_1 \sin\gamma_2) \cdot f_{\mathcal{G}}(\gamma_1) f_{\mathcal{G}}(\gamma_2)\, d\gamma_1\, d\gamma_2 \quad (91)$$

such that

$$\boxed{\sigma_\gamma^c = (\mu_\gamma^c)^2 + (\mu_\gamma^s)^2, \qquad \sigma_\gamma^s = \mu_\gamma^s \mu_\gamma^c - \mu_\gamma^c \mu_\gamma^s = 0}. \quad (92)$$

The matrix $\overline{\mathbf{Q}}_{\mathrm{tot}}$ can now be easily computed as

$$\boxed{\overline{\mathbf{Q}}_{\mathrm{tot}} = \begin{bmatrix} \sigma_a^2 \mathbf{1}_L & \mu_a^2\sigma_\gamma^c \mathbf{1}_L & \cdots & \mu_a^2\sigma_\gamma^c \mathbf{1}_L \\ \mu_a^2\sigma_\gamma^c \mathbf{1}_L & \sigma_a^2 \mathbf{1}_L & \cdots & \mu_a^2\sigma_\gamma^c \mathbf{1}_L \\ \vdots & \vdots & & \vdots \\ \mu_a^2\sigma_\gamma^c \mathbf{1}_L & \mu_a^2\sigma_\gamma^c \mathbf{1}_L & \cdots & \sigma_a^2 \mathbf{1}_L \end{bmatrix} \odot Q_{LS}} \quad (93)$$

where $\mathbf{1}_L$ is an $L \times L$-dimensional matrix with all elements equal to 1 and $\odot$ denoting element-wise multiplication. As can be seen, we only need the mean and the variance of the gain pdf $f_\alpha(a)$, whereas in general, complete knowledge of the phase pdf $f_{\mathcal{G}}(\gamma)$ is required.

Frequently used probability density functions are a uniform distribution (with boundary values $a_{\min}$ and $a_{\max}$)

$$\begin{cases} f_\alpha(a) = \dfrac{1}{a_{\max} - a_{\min}}, & a_{\min} \leq a \leq a_{\max} \\ \quad\;\; = 0, & a < a_{\min},\, a > a_{\max} \end{cases} \quad (94)$$

and a Gaussian distribution (with mean μ_a and standard deviation s_a)

$$f_\alpha(a) = \frac{1}{\sqrt{2\pi s_a^2}}\, e^{-((a-\mu_a)^2/2s_a^2)}. \quad (95)$$

For a uniform distribution, the different gain and phase parameters are equal to

$$\mu_a = \frac{a_{\min} + a_{\max}}{2} \qquad \sigma_a^2 = \frac{a_{\min}^2 + a_{\min}a_{\max} + a_{\max}^2}{3}$$
$$\mu_\gamma^c = \frac{\sin\gamma_{\max} - \sin\gamma_{\min}}{\gamma_{\max} - \gamma_{\min}} \qquad \mu_\gamma^s = \frac{\cos\gamma_{\min} - \cos\gamma_{\max}}{\gamma_{\max} - \gamma_{\min}}$$
$$\sigma_\gamma^c = \frac{2 - 2\cos(\gamma_{\max} - \gamma_{\min})}{(\gamma_{\max} - \gamma_{\min})^2} \qquad \sigma_\gamma^s = 0.$$

For a Gaussian distribution with mean μ_a and standard deviation s_a, the variance is equal to $\sigma_a^2 = \mu_a^2 + s_a^2$, whereas the phase parameters μ_γ^c, μ_γ^s, and σ_γ^c have to be calculated numerically.

2) Nonlinear Cost Function: The mean performance nonlinear cost function can be written as

$$J_{\mathrm{NL}}^{tot}(\mathbf{w}) = \int_{A_0} \cdots \int_{A_{N-1}} J_{\mathrm{NL}}(\mathbf{w}, A_0, \ldots, A_{N-1}) \cdot f_{\mathcal{A}}(A_0) \cdots f_{\mathcal{A}}(A_{N-1})\, dA_0 \cdots dA_{N-1}. \quad (96)$$

The cost function $J_{\mathrm{NL}}(\mathbf{w}, A_0, \ldots, A_{N-1})$ for a specific microphone characteristic is equal to (46), i.e.,

$$J_{\mathrm{NL}}(\mathbf{w}, A_0, \ldots, A_{N-1}) = \overline{J}_{sum}(\mathbf{w}) + d_{\mathrm{NL}} - 2\mathbf{w}^T \overline{\mathbf{Q}}_{\mathrm{NL}} \mathbf{w}. \quad (97)$$

By combining (96) and (97), the mean performance nonlinear cost function can be written as

$$J_{\mathrm{NL}}^{tot}(\mathbf{w}) = \int_{A_0} \cdots \int_{A_{N-1}} \overline{J}_{sum}(\mathbf{w}) \cdot f_{\mathcal{A}}(A_0) \cdots f_{\mathcal{A}}(A_{N-1})\, dA_0 \cdots dA_{N-1} - 2\mathbf{w}^T \int_{A_0} \cdots \int_{A_{N-1}} \overline{\mathbf{Q}}_{\mathrm{NL}} \cdot f_{\mathcal{A}}(A_0) \cdots f_{\mathcal{A}}(A_{N-1})\, dA_0 \cdots dA_{N-1} \mathbf{w} + \int_{A_0} \cdots \int_{A_{N-1}} d_{\mathrm{NL}} \cdot f_{\mathcal{A}}(A_0) \cdots f_{\mathcal{A}}(A_{N-1})\, dA_0 \cdots dA_{N-1} \quad (98)$$

$$= \overline{J}_{sum}^{\,tot}(\mathbf{w}) - 2\mathbf{w}^T \overline{\mathbf{Q}}_{NL}^{\,tot} \mathbf{w} + d_{\mathrm{NL}}. \quad (99)$$

Similarly to (93), the matrix $\overline{\mathbf{Q}}_{NL}^{tot}$ is equal to

$$\boxed{\overline{\mathbf{Q}}_{NL}^{tot} = \begin{bmatrix} \sigma_a^2\mathbf{1}_L & \mu_a^2\sigma_\gamma^c\mathbf{1}_L & \cdots & \mu_a^2\sigma_\gamma^c\mathbf{1}_L \\ \mu_a^2\sigma_\gamma^c\mathbf{1}_L & \sigma_a^2\mathbf{1}_L & \cdots & \mu_a^2\sigma_\gamma^c\mathbf{1}_L \\ \vdots & \vdots & & \vdots \\ \mu_a^2\sigma_\gamma^c\mathbf{1}_L & \mu_a^2\sigma_\gamma^c\mathbf{1}_L & \cdots & \sigma_a^2\mathbf{1}_L \end{bmatrix} \odot \mathbf{Q}_{\mathrm{NL}}}. \tag{100}$$

Using (58), $\overline{J}_{sum}^{tot}(\mathbf{w})$ can be written as

$$\begin{aligned} \overline{J}_{sum}^{tot}(\mathbf{w}) &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{k=1}^{M}\sum_{l=1}^{M} w(i)w(j)w(k)w(l) \\ &\quad \cdot \int_{A_0}\cdots\int_{A_{N-1}} a_{ijkl}(\cos\gamma_{ijkl}\cdot\rho_{ijkl} - \sin\gamma_{ijkl}\cdot\rho_{ijkl}^{\circ}) \\ &\quad \cdot f_{\mathcal{A}}(A_0)\cdots f_{\mathcal{A}}(A_{N-1})\,dA_0\cdots dA_{N-1} \qquad (101) \\ &= \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{k=1}^{M}\sum_{l=1}^{M} w(i)w(j)w(k)w(l)\delta_{ijkl}^{a} \\ &\quad \cdot (\delta_{\gamma,ijkl}^{c}\cdot\rho_{ijkl} - \delta_{\gamma,ijkl}^{s}\cdot\rho_{ijkl}^{\circ}) \qquad (102) \end{aligned}$$

where

$$\begin{aligned} \delta_{ijkl}^{a} &= \int_{a_0}\cdots\int_{a_{N-1}} a_{ijkl}f_\alpha(a_0)\cdots f_\alpha(a_{N-1})\,da_0\cdots da_{N-1} \qquad (103) \\ \delta_{\gamma,ijkl}^{c} &= \int_{\gamma_0}\cdots\int_{\gamma_{N-1}} \cos\gamma_{ijkl} f_{\mathcal{G}}(\gamma_0)\cdots f_{\mathcal{G}}(\gamma_{N-1})\,d\gamma_0\cdots d\gamma_{N-1} \qquad (104) \\ \delta_{\gamma,ijkl}^{s} &= \int_{\gamma_0}\cdots\int_{\gamma_{N-1}} \sin\gamma_{ijkl} f_{\mathcal{G}}(\gamma_0)\cdots f_{\mathcal{G}}(\gamma_{N-1})\,d\gamma_0\cdots d\gamma_{N-1} \qquad (105) \end{aligned}$$

where a_{ijkl} and γ_{ijkl} are defined in (56). The expression $\overline{J}_{sum}^{tot}(\mathbf{w})$ in (102) has the same form as (58), such that the same nonlinear optimization techniques as described in Section III-C3 can be used for minimizing $J_{\mathrm{NL}}^{tot}(\mathbf{w})$. The calculation of the parameters δ_{ijkl}^{a}, $\delta_{\gamma,ijkl}^{c}$ and $\delta_{\gamma,ijkl}^{s}$ is discussed in Appendix B. For the calculation of δ_{ijkl}^{a}, we only require the (higher order) moments of the gain pdf $f_\alpha(a)$, whereas for the calculation of $\delta_{\gamma,ijkl}^{c}$ and $\delta_{\gamma,ijkl}^{s}$, in general, complete knowledge of the phase pdf $f_{\mathcal{G}}(\gamma)$ is required. In Appendix B, it is also shown that for a symmetric phase pdf, $\delta_{\gamma,ijkl}^{s} = 0$, such that

$$\boxed{\overline{J}_{sum}^{tot}(\mathbf{w}) = \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{k=1}^{M}\sum_{l=1}^{M} w(i)w(j)w(k)w(l)\,\delta_{ijkl}^{a}\cdot\delta_{\gamma,ijkl}^{c}\cdot\rho_{ijkl}}. \tag{106}$$

B. Minimax Criterion

For the minimax criterion, which optimizes the worst-case performance, we first have to define a (finite) set of microphone characteristics (K_a gain values and K_γ phase values)

$$\begin{aligned} &\{a_{\min} = a_1,\, a_2,\, \ldots,\, a_{K_a} = a_{\max}\} \\ &\{\gamma_{\min} = \gamma_1,\, \gamma_2,\, \ldots,\, \gamma_{K_\gamma} = \gamma_{\max}\} \end{aligned} \tag{107}$$

as an approximation for the continuum of feasible microphone characteristics and use this set to construct the $(K_aK_\gamma)^N$-dimensional vector $\mathbf{F}(\mathbf{w})$

$$\mathbf{F}(\mathbf{w}) = \begin{bmatrix} F_1(\mathbf{w}, A_0, \ldots, A_{N-1}) \\ F_2(\mathbf{w}, A_0, \ldots, A_{N-1}) \\ \vdots \\ F_{(K_aK_\gamma)^N}(\mathbf{w}, A_0, \ldots, A_{N-1}) \end{bmatrix} \tag{108}$$

which consists of the used cost function (weighted LS, TLS eigenfilter, nonlinear, or any other cost function, e.g., defined in [19]–[23]) at each possible combination of gain and phase values. The goal then is to minimize the L_∞-norm of $\mathbf{F}(\mathbf{w})$, i.e., the maximum value of the elements $F_k(\mathbf{w})$

$$\boxed{\min_{\mathbf{w}} \|\mathbf{F}(\mathbf{w})\|_\infty = \min_{\mathbf{w}} \max_{k} F_k(\mathbf{w})} \tag{109}$$

which can, e.g., be done using a sequential quadratic programming (SQP) method [36], [37]. In order to improve the numerical robustness and the convergence speed, the gradient

$$\left[\frac{\partial F_1(\mathbf{w}, A_0, \ldots, A_{N-1})}{\partial \mathbf{w}} \quad \frac{\partial F_2(\mathbf{w}, A_0, \ldots, A_{N-1})}{\partial \mathbf{w}} \quad \cdots \quad \frac{\partial F_{(K_aK_\gamma)^N}(\mathbf{w}, A_0, \ldots, A_{N-1})}{\partial \mathbf{w}} \right] \tag{110}$$

which is an $M \times (K_aK_\gamma)^N$-dimensional matrix, can be supplied analytically. As can easily be seen, the larger the values K_a and K_γ, the denser the grid of feasible microphone characteristics, and the higher the computational complexity for solving the minimax problem. However, when only considering gain errors and using the weighted LS cost function, the number of grid points can be drastically reduced.

Theorem 1: When considering only **gain errors** and using the **weighted LS cost function**, the maximum value of $\mathbf{F}(\mathbf{w})$, for any $\mathbf{w}$, occurs on a boundary point (of an N-dimensional hypercube), i.e., $a_n = a_{\min}$ or $a_n = a_{\max}$, $n = 0\cdots N-1$. This implies that $K_a = 2$ suffices, and $\mathbf{F}(\mathbf{w})$ only consists of 2^N elements. This is not necessarily the case for the TLS eigenfilter and the nonlinear cost function.

Proof: When considering only gain errors, the weighted LS cost function in (74) can be written as

$$J_{LS}(\mathbf{w}, a_0, \ldots, a_{N-1}) = \mathbf{w}^T\overline{\mathbf{Q}}_{LS}\mathbf{w} - 2\mathbf{w}^T\overline{\mathbf{a}} + d_{LS} \tag{111}$$

where $\overline{\mathbf{a}} = \mathbf{A}_R\mathbf{a}$ and $\overline{\mathbf{Q}}_{LS} = \mathbf{A}_R\mathbf{Q}_{LS}\mathbf{A}_R$, and

$$\mathbf{A}_R = \begin{bmatrix} a_0\mathbf{I}_L & & & \\ & a_1\mathbf{I}_L & & \\ & & \ddots & \\ & & & a_{N-1}\mathbf{I}_L \end{bmatrix}. \tag{112}$$

The expression $\mathbf{w}^T\overline{\mathbf{Q}}_{LS}\mathbf{w}$ can be rewritten as

$$\mathbf{w}^T\overline{\mathbf{Q}}_{LS}\mathbf{w} = \mathbf{w}^T\mathbf{A}_R\mathbf{Q}_{LS}\mathbf{A}_R\mathbf{w} = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_i a_j \mathbf{w}_i^T\mathbf{Q}_{LS}(i,\, j)\mathbf{w}_j \quad (113)$$

where $\mathbf{Q}_{LS}(i,\, j)$ is an $L \times L$-dimensional submatrix of $\mathbf{Q}_{LS}$, i.e.,

$$\mathbf{Q}_{LS}(i,\, j) = \mathbf{Q}_{LS}^{iL+1:(i+1)L,\, jL+1:(j+1)L} \qquad i = 0\cdots N-1,\, j = 0\cdots N-1. \quad (114)$$

If we substitute $\mathbf{w}_i^T\mathbf{Q}_{LS}(i,\, j)\mathbf{w}_j$ into $b_{ij}(\mathbf{w})$, then $\mathbf{w}^T\overline{\mathbf{Q}}_{LS}\mathbf{w}$ in (113) can be rewritten as

$$\mathbf{w}^T\overline{\mathbf{Q}}_{LS}\mathbf{w} = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_i a_j b_{ij}(\mathbf{w}) = \boldsymbol{\alpha}^T\mathbf{B}_{LS}(\mathbf{w})\boldsymbol{\alpha} \quad (115)$$

where $\boldsymbol{\alpha}$ is an N-dimensional vector consisting of the microphone gains

$$\boldsymbol{\alpha} = [\,a_0 \quad a_1 \quad \cdots \quad a_{N-1}\,]^T. \quad (116)$$

Similarly, if we define $c_i(\mathbf{w})$ as $\mathbf{w}_i^T\mathbf{a}(i)$, where $\mathbf{a}(i)$ is an L-dimensional subvector of $\mathbf{a}$, $\mathbf{a}(i) = \mathbf{a}^{iL+1:(i+1)L}$, and $i = 0\cdots N-1$, then the weighted LS cost function can be written as

$$J_{LS}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}^T\mathbf{B}_{LS}(\mathbf{w})\boldsymbol{\alpha} - 2\boldsymbol{\alpha}^T\mathbf{c} + d_{LS}. \quad (117)$$

Since $\mathbf{Q}_{LS}$ is a positive-(semi)definite matrix, $\mathbf{w}^T\mathbf{Q}_{LS}\mathbf{w} \geq 0$, $\forall\, \mathbf{w}$ such that

$$\mathbf{w}^T\overline{\mathbf{Q}}_{LS}\mathbf{w} = \mathbf{w}^T\mathbf{A}_R\mathbf{Q}_{LS}\mathbf{A}_R\mathbf{w} = \boldsymbol{\alpha}^T\mathbf{B}_{LS}(\mathbf{w})\boldsymbol{\alpha} \geq 0, \qquad \forall\,\mathbf{w},\, \forall\,\boldsymbol{\alpha} \quad (118)$$

and $\mathbf{B}_{LS}(\mathbf{w})$ is a positive-(semi)definite matrix for every $\mathbf{w}$. Therefore, the weighted LS cost function $J_{LS}(\boldsymbol{\alpha})$ is a quadratic function (with a single minimum), such that the maximum value of $J_{LS}(\boldsymbol{\alpha})$ for all points inside an N-dimensional hypercube, defined by $a_{\min} \leq a_n \leq a_{\max}$, $n = 0\cdots N-1$, occurs on one of the 2^N boundary points of the hypercube. ■

V. Simulations

This section discusses the simulation results of robust broadband beamformer design for gain and phase errors in the microphone characteristics. Since the effect of gain and phase errors is more profound for small-size microphone arrays, we have performed simulations for a small-size linear nonuniform microphone array consisting of $N = 3$ microphones at positions $[-0.01 \quad 0 \quad 0.015]$ m, corresponding to a typical configuration for a next-generation multimicrophone BTE hearing aid. The nominal gains and phases of the microphones are $a_n = 1$ and $\gamma_n = 0°$, $n = 0\cdots N-1$. We have designed an end-fire broadband beamformer for a sampling frequency $f_s = 8$ kHz with passband specifications $(\Omega_p,\, \Theta_p) = $ (300–4000 Hz, 0°–60°) and stopband specifications $(\Omega_s,\, \Theta_s) = $ (300–4000 Hz, 80°–180°), cf. Sections III-A3 and C4. For the TLS eigenfilter, the matrix $\overline{\mathbf{Q}}_e^{tot}$ is computed with frequency and angle specifications $(\Omega,\, \Theta) = $ (300–4000 Hz, 0°–180°), cf. Section III-B. The used filter length $L = 20$, and the stopband weight $\alpha = 1$.

TABLE I
Different Cost Functions for Weighted LS, TLS Eigenfilter, and Nonlinear Robust Beamformer Design ($\alpha = 1$; $N = 3$; $L = 20$)

Design procedure		*Cost functions*				
		J	J_a^{tot}	J_γ^{tot}	J_A^{tot}	J_{max}
LS	Non-robust	**0.3125**	123.28	62.674	185.65	961.33
LS	Gain	0.4740	**0.6419**	0.5759	0.7438	1.4409
LS	Phase	0.4313	0.6660	**0.5566**	0.7913	1.7488
LS	Gain-phase	0.5176	0.6531	0.5961	**0.7315**	1.3680
LS	Minimax	0.7466	0.8035	0.7920	0.8490	**1.0346**
NL	Non-robust	**0.1585**	124.56	70.194	275.40	3623.6
NL	Gain	0.1762	**0.2181**	0.3389	0.3926	0.5047
NL	Phase	0.2072	0.2594	**0.3001**	0.3568	0.5018
NL	Gain-phase	0.2186	0.2481	0.3044	**0.3371**	0.4990
NL	Minimax	0.1707	0.2300	0.3402	0.4114	**0.4167**
TLS	Non-robust	**0.0748**				0.9356
TLS	Minimax	0.1670				**0.2460**

We have designed several types of beamformers using the weighted LS cost function and the nonlinear criterion:

1) a nonrobust broadband beamformer (not taking into account errors, i.e., assuming $a_n = 1$, $\gamma_n = 0°$);
2) a robust broadband beamformer using a uniform gain pdf ($a_{\min} = 0.85$, $a_{\max} = 1.15$);
3) a robust broadband beamformer using a uniform phase pdf ($\gamma_{\min} = -5°$, $\gamma_{\max} = 10°$);
4) a robust broadband beamformer using a uniform gain/phase pdf ($a_{\min} = 0.85$, $a_{\max} = 1.15$, $\gamma_{\min} = -5°$, $\gamma_{\max} = 10°$);
5) a robust broadband beamformer using the minimax criterion (only gain errors are taken into account, $a_{\min} = 0.85$, $a_{\max} = 1.15$, $K_a = 5$).

Using the TLS eigenfilter cost function, we have designed a nonrobust beamformer and a robust beamformer using the minimax criterion. For all beamformer designs, we have computed the following cost functions:

1) the cost function J without phase and gain errors ($a_n = 1$, $\gamma_n = 0°$);
2) the mean cost function J_a^{tot} for the uniform gain pdf;
3) the mean cost function J_γ^{tot} for the uniform phase pdf;
4) the mean cost function J_A^{tot} for the uniform gain/phase pdf;
5) the maximum cost function $J_{\max}$ when the gain varies between $a_{\min} = 0.85$ and $a_{\max} = 1.15$.

We will plot the spatial directivity pattern in the frequency-angle region (300–3500 Hz, 0°–180°) and the angular pattern for the specific frequencies (500, 1000, 1500, 2000, 2500, 3500) Hz.

Table I summarizes the different cost functions for the weighted LS, the nonlinear, and the TLS eigenfilter nonrobust

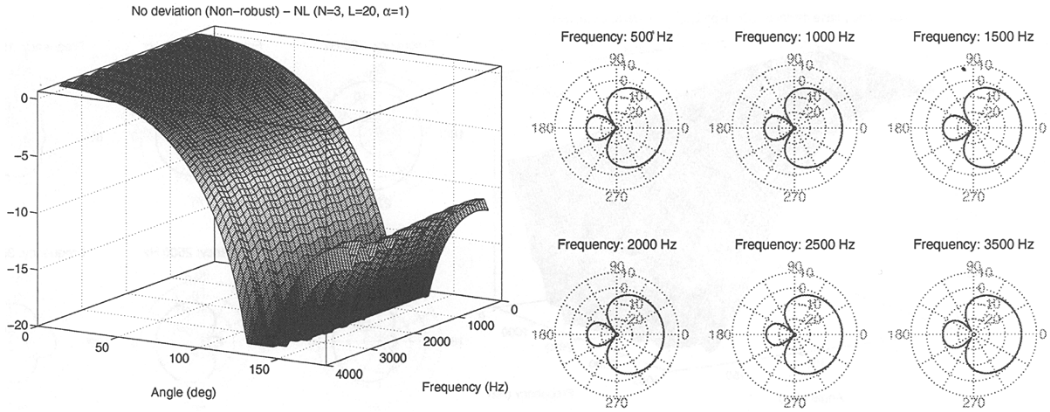

Fig. 2. Spatial directivity pattern of nonlinear nonrobust design for no gain and phase errors ($\alpha = 1, N = 3, L = 20$).

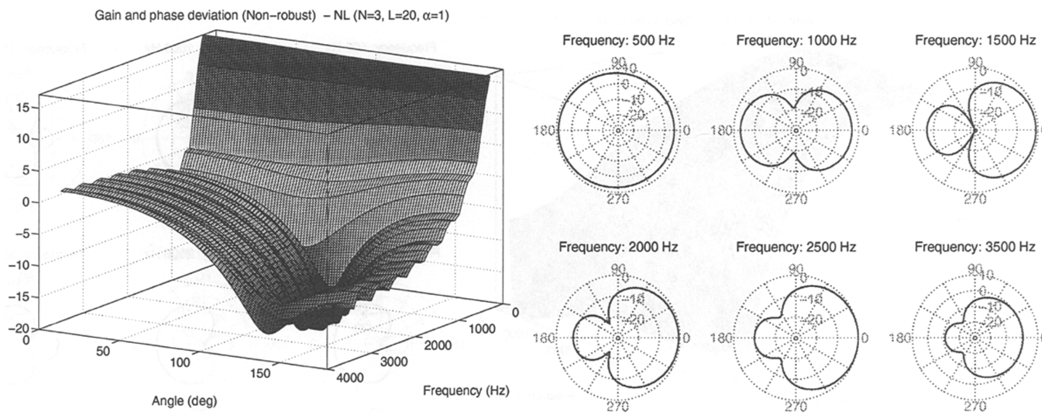

Fig. 3. Spatial directivity pattern of nonlinear nonrobust design for gain and phase errors ($\alpha = 1, N = 3, L = 20$).

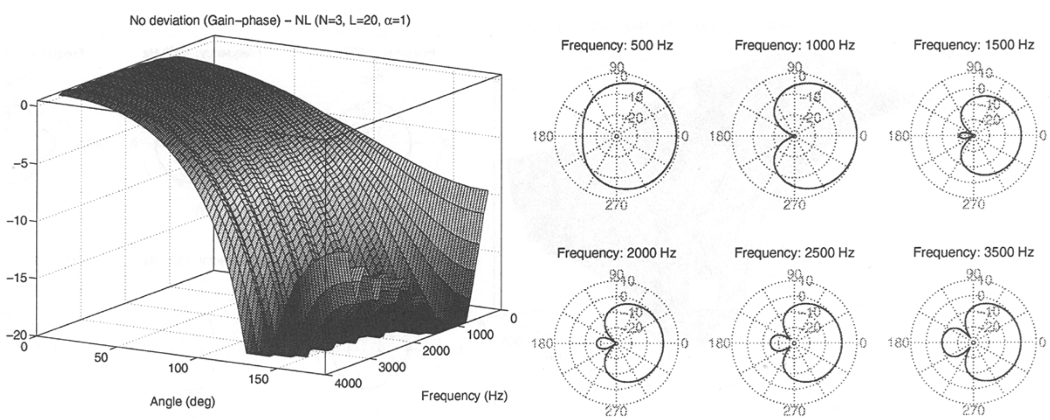

Fig. 4. Spatial directivity pattern of nonlinear gain/phase-robust design for no gain and phase errors ($\alpha = 1, N = 3, L = 20$).

and robust broadband beamformer design procedures. Obviously, the design procedure optimizing a specific cost function leads to the best value for this cost function (bold values). This implies that when no gain and phase errors occur, the robust design procedures lead to a higher cost function J than the nonrobust design procedure. However, the nonrobust design procedure leads to very poor results whenever gain and/or phase errors occur (e.g., compare $J_{\max}$ for the nonrobust and the robust design procedures and see the figures). All robust design procedures (using pdf and minimax criterion) yield satisfactory results when gain and/or phase errors occur.

Fig. 2 shows the spatial directivity pattern of the nonrobust beamformer, designed with the nonlinear cost function, when no gain and phase errors occur. Fig. 3 shows the spatial directivity pattern for microphone gains $[0.9 \;\; 1.1 \;\; 1.05]$ and phases $[5^\circ \;\; -2^\circ \;\; 5^\circ]$, i.e., small deviations from the nominal gains and phases. As can be seen from this figure, the beamformer performance dramatically degrades, especially for the lower frequen-

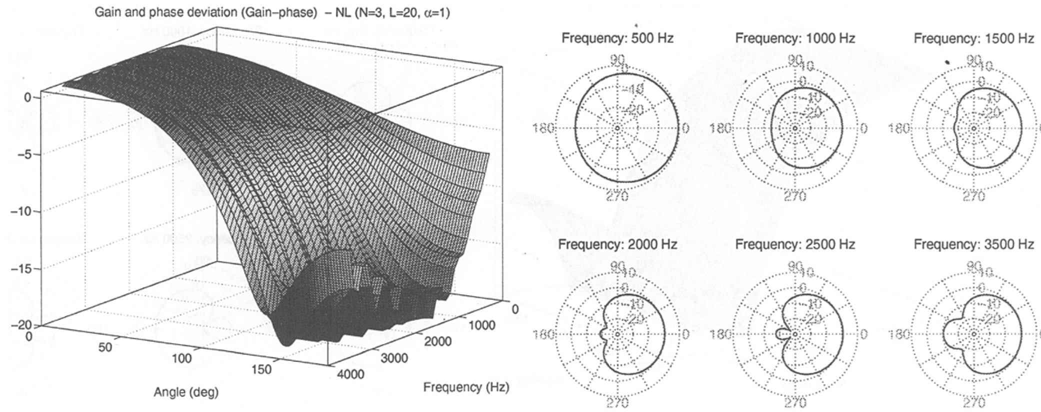

Fig. 5. Spatial directivity pattern of nonlinear gain/phase-robust design for gain and phase errors ($\alpha = 1, N = 3, L = 20$).

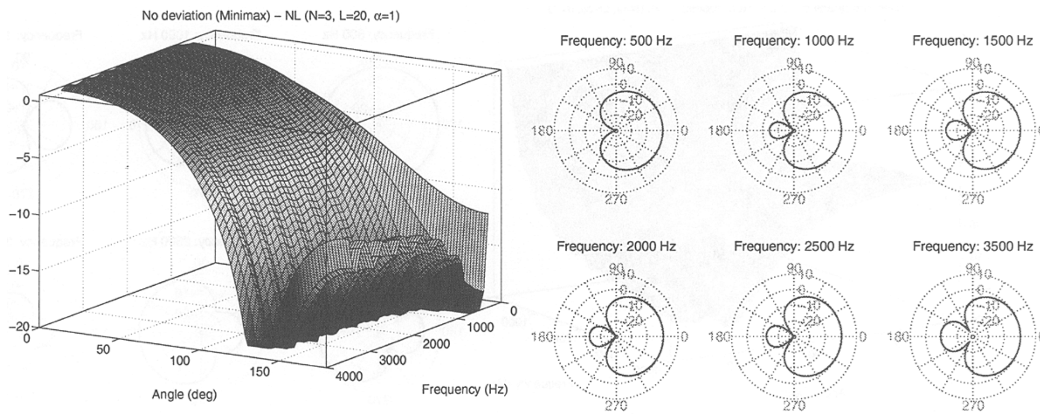

Fig. 6. Spatial directivity pattern of nonlinear minimax design for no gain and phase errors ($\alpha = 1, N = 3, L = 20$).

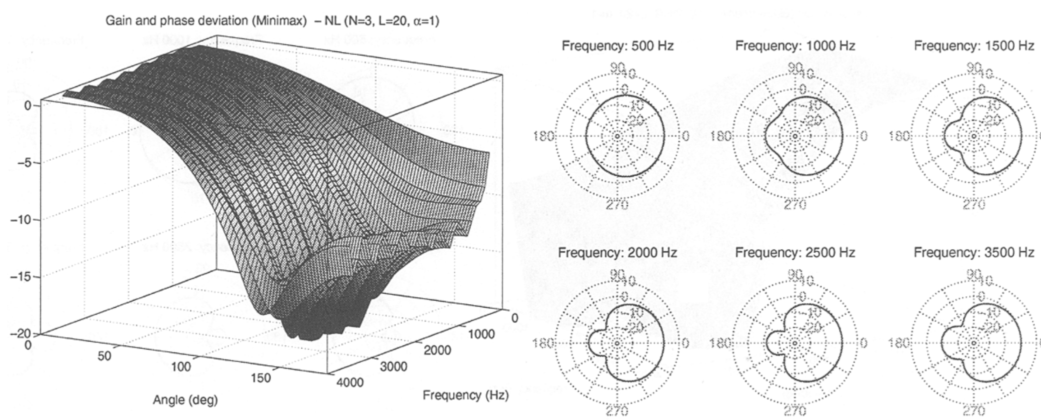

Fig. 7. Spatial directivity pattern of nonlinear minimax design for gain and phase errors ($\alpha = 1, N = 3, L = 20$).

cies, where the spatial directivity pattern is almost omni-directional, and the amplification is very high.

Figs. 4 and 5 show the spatial directivity pattern of the gain/phase-robust beamformer, designed with the nonlinear cost function, when no errors occur and when gain and phase errors occur. As can be seen from Fig. 4, the performance of this beamformer is worse than the performance of the nonrobust beamformer when no errors occur. However, as can be clearly seen from Fig. 5, when gain and phase errors occur, the performance of the gain/phase-robust beamformer is much better than the performance of the nonrobust beamformer.

Figs. 6 and 7 show the spatial directivity pattern of the minimax beamformer, designed with the nonlinear cost function, when no errors occur and when gain and phase errors occur. Similar conclusions can be drawn for the minimax beamformer as for the gain/phase-robust beamformer.

VI. Conclusions

In this paper, two procedures have been presented for designing fixed broadband beamformers that are robust against gain and phase errors in the microphone array characteristics.

The first design procedure optimizes the mean performance by minimizing a weighted sum using the gain and the phase probability density functions. The second design procedure optimizes the worst-case performance by minimizing the maximum cost function over a finite set of feasible microphone characteristics. We have used the weighted LS, the TLS eigenfilter, and a nonlinear cost function for designing broadband beamformers with an arbitrary spatial directivity pattern. Simulation results for the different design procedures and cost functions show that robust broadband beamformer design for a small-size microphone array indeed leads to a significant performance improvement when gain and phase errors occur.

Appendix A
Calculation of Double Integral for Far-Field

The integral

$$I = \int_{\theta_1}^{\theta_2} \int_{\omega_1}^{\omega_2} \cos[\omega(\alpha + \beta\cos\theta) + \gamma]\, d\omega\, d\theta \tag{119}$$

is equal to

$$\int_{\theta_1}^{\theta_2} \frac{\sin[\omega_2(\alpha + \beta\cos\theta) + \gamma]}{\alpha + \beta\cos\theta}\, d\theta - \int_{\theta_1}^{\theta_2} \frac{\sin[\omega_1(\alpha + \beta\cos\theta) + \gamma]}{\alpha + \beta\cos\theta}\, d\theta \tag{120}$$

such that, in fact, we need to solve integrals of the type (120)

$$I_\theta(\omega) = \int_{\theta_1}^{\theta_2} f(\omega,\, \theta)\, d\theta \tag{121}$$

where

$$\boxed{f(\omega,\, \theta) = \frac{\sin[\omega(\alpha + \beta\cos\theta) + \gamma]}{\alpha + \beta\cos\theta}}. \tag{122}$$

Normally, this integral can be solved numerically without any problem, but a special case occurs when $|\alpha| \leq |\beta|$ because then, a singularity θ_n occurs in the denominator, with

$$\cos\theta_n = -\frac{\alpha}{\beta} \tag{123}$$

such that numerically calculating the integral $I_\theta(\omega)$ could lead to numerical problems when $\gamma \neq 0$. By using the Taylor expansion of $\cos\theta$ around θ_n, we can derive a function $g(\theta)$

$$g(\theta) = -\frac{\sin\gamma}{\beta\sqrt{1 - \frac{\alpha^2}{\beta^2}}(\theta - \theta_n)} \tag{124}$$

which is a good approximation for $f(\omega,\, \theta)$ around θ_n and which is independent of ω. If we now define the function $\overline{f}(\omega,\, \theta) = f(\omega,\, \theta) - g(\theta)$, we can prove (by applying L'Hôpital's rule twice) that for any γ, $\lim_{\theta\to\theta_n} \overline{f}(\omega,\, \theta)$ is finite and is equal to

$$\lim_{\theta\to\theta_n} \overline{f}(\omega,\, \theta) = \omega\cos\gamma + \frac{\alpha\sin\gamma}{2(\alpha^2 - \beta^2)}. \tag{125}$$

For details, see [18].

Hence, the function $\overline{f}(\omega,\, \theta)$ can be integrated numerically with no problem. In fact, the total integral I can be written as

$$\begin{aligned} I &= I_\theta(\omega_2) - I_\theta(\omega_1) \\ &= \int_{\theta_1}^{\theta_2} f(\omega_2,\, \theta)\, d\theta - \int_{\theta_1}^{\theta_2} f(\omega_1,\, \theta)\, d\theta &(126) \\ &= \int_{\theta_1}^{\theta_2} \overline{f}(\omega_2,\, \theta)\, d\theta - \int_{\theta_1}^{\theta_2} \overline{f}(\omega_1,\, \theta)\, d\theta. &(127) \end{aligned}$$

Appendix B
Calculation of δ^a_{ijkl}, $\delta^c_{\gamma,\,ijkl}$ and $\delta^s_{\gamma,\,ijkl}$ for Robust Nonlinear Criterion

Depending on the values of $\lfloor(i-1)/L\rfloor$, $\lfloor(j-1)/L\rfloor$, $\lfloor(k-1)/L\rfloor$, and $\lfloor(l-1)/L\rfloor$, different cases have to be considered:

- **Four equal values:** $\gamma_{ijkl} = 0$

$$\delta^a_{ijkl} = \int_a a^4 f_\alpha(a)\, da, \quad \delta^c_{\gamma,\,ijkl} = 1, \quad \delta^s_{\gamma,\,ijkl} = 0. \tag{128}$$

- **Three equal values and one different value:** $a_{ijkl} = a_1^3 a_2$, $\gamma_{ijkl} = \pm(\gamma_1 - \gamma_2)$

$$\begin{aligned} \delta^a_{ijkl} &= \int_{a_1}\int_{a_2} a_1^3 a_2 f_\alpha(a_1) f_\alpha(a_2)\, da_1\, da_2 \\ &= \mu_a \int_a a^3 f_\alpha(a)\, da &(129) \\ \delta^c_{\gamma,\,ijkl} &= \int_{\gamma_1}\int_{\gamma_2} \cos[\pm(\gamma_1 - \gamma_2)] f_{\mathcal{G}}(\gamma_1) f_{\mathcal{G}}(\gamma_2)\, d\gamma_1\, d\gamma_2 \\ &= \sigma^c_\gamma &(130) \\ \delta^s_{\gamma,\,ijkl} &= \int_{\gamma_1}\int_{\gamma_2} \sin[\pm(\gamma_1 - \gamma_2)] f_{\mathcal{G}}(\gamma_1) f_{\mathcal{G}}(\gamma_2)\, d\gamma_1\, d\gamma_2 \\ &= 0. &(131) \end{aligned}$$

- **Two equal values and two equal values:** $a_{ijkl} = a_1^2 a_2^2$

$$\begin{aligned} \delta^a_{ijkl} &= \int_{a_1}\int_{a_2} a_1^2 a_2^2 f_\alpha(a_1) f_\alpha(a_2)\, da_1\, da_2 \\ &= \sigma_a^4 &(132) \end{aligned}$$

$$\boxed{\lfloor\tfrac{i-1}{L}\rfloor = \lfloor\tfrac{k-1}{L}\rfloor \neq \lfloor\tfrac{j-1}{L}\rfloor = \lfloor\tfrac{l-1}{L}\rfloor} \;:\; \gamma_{ijkl} = 2(\gamma_1 - \gamma_2)$$

$$\begin{aligned} \delta^c_{\gamma,\,ijkl} &= \int_{\gamma_1}\int_{\gamma_2} \cos 2(\gamma_1 - \gamma_2) f_{\mathcal{G}}(\gamma_1) \\ &\quad \cdot f_{\mathcal{G}}(\gamma_2)\, d\gamma_1\, d\gamma_2 \\ &= \int_{\gamma_1}\int_{\gamma_2} (\cos 2\gamma_1 \cos 2\gamma_2 + \sin 2\gamma_1 \sin 2\gamma_2) \\ &\quad \cdot f_{\mathcal{G}}(\gamma_1) f_{\mathcal{G}}(\gamma_2)\, d\gamma_1\, d\gamma_2 \\ &= (\mu^c_{2\gamma})^2 + (\mu^s_{2\gamma})^2 &(133) \\ \delta^s_{\gamma,\,ijkl} &= \int_{\gamma_1}\int_{\gamma_2} \sin 2(\gamma_1 - \gamma_2) f_{\mathcal{G}}(\gamma_1) f_{\mathcal{G}}(\gamma_2)\, d\gamma_1\, d\gamma_2 \\ &= 0 &(134) \end{aligned}$$

where

$$\mu^c_{2\gamma} = \int_\gamma \cos 2\gamma f_{\mathcal{G}}(\gamma)\, d\gamma, \quad \mu^s_{2\gamma} = \int_\gamma \sin 2\gamma f_{\mathcal{G}}(\gamma)\, d\gamma \tag{135}$$

$$\boxed{\begin{array}{l}\lfloor\frac{i-1}{L}\rfloor=\lfloor\frac{j-1}{L}\rfloor\neq\lfloor\frac{k-1}{L}\rfloor=\lfloor\frac{l-1}{L}\rfloor,\\ \lfloor\frac{i-1}{L}\rfloor=\lfloor\frac{l-1}{L}\rfloor\neq\lfloor\frac{j-1}{L}\rfloor=\lfloor\frac{k-1}{L}\rfloor\end{array}}\;:\;\gamma_{ijkl}=0$$

$$\delta^c_{\gamma,\,ijkl}=1,\qquad \delta^s_{\gamma,\,ijkl}=0. \tag{136}$$

- **Two equal values and two different values:** $a_{ijkl} = a_1^2 a_2 a_3$

$$\begin{aligned}\delta^a_{ijkl} &= \int_{a_1}\int_{a_2}\int_{a_3} a_1^2 a_2 a_3 f_\alpha(a_1) f_\alpha(a_2) f_\alpha(a_3)\, da_1\, da_2\, da_3\\ &= \sigma_a^2\mu_a^2 \end{aligned}\tag{137}$$

$$\boxed{\lfloor\tfrac{i-1}{L}\rfloor=\lfloor\tfrac{k-1}{L}\rfloor\neq\lfloor\tfrac{j-1}{L}\rfloor\neq\lfloor\tfrac{l-1}{L}\rfloor}\;:\;\gamma_{ijkl}=2\gamma_1-\gamma_2-\gamma_3$$

$$\begin{aligned}\delta^c_{\gamma,\,ijkl} &= \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3}\cos(2\gamma_1-\gamma_2-\gamma_3)\\ &\quad\cdot f_{\mathcal{G}}(\gamma_1)f_{\mathcal{G}}(\gamma_2)f_{\mathcal{G}}(\gamma_3)\,d\gamma_1\,d\gamma_2\,d\gamma_3\\ &= \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3}\cos 2\gamma_1(\cos\gamma_2\cos\gamma_3-\sin\gamma_2\sin\gamma_3)\\ &\quad+\sin 2\gamma_1(\sin\gamma_2\cos\gamma_3+\cos\gamma_2\sin\gamma_3)\\ &\quad\cdot f_{\mathcal{G}}(\gamma_1)f_{\mathcal{G}}(\gamma_2)f_{\mathcal{G}}(\gamma_3)\,d\gamma_1\,d\gamma_2\,d\gamma_3\\ &= \mu^c_{2\gamma}\left[(\mu^c_\gamma)^2-(\mu^s_\gamma)^2\right]+2\mu^s_{2\gamma}\mu^c_\gamma\mu^s_\gamma=\overline{\delta}^{\,c}_\gamma \end{aligned}\tag{138}$$

$$\begin{aligned}\delta^s_{\gamma,\,ijkl} &= \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3}\sin(2\gamma_1-\gamma_2-\gamma_3)\\ &\quad\cdot f_{\mathcal{G}}(\gamma_1)f_{\mathcal{G}}(\gamma_2)f_{\mathcal{G}}(\gamma_3)\,d\gamma_1\,d\gamma_2\,d\gamma_3\\ &= \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3}\sin 2\gamma_1(\cos\gamma_2\cos\gamma_3-\sin\gamma_2\sin\gamma_3)\\ &\quad-\cos 2\gamma_1(\sin\gamma_2\cos\gamma_3+\cos\gamma_2\sin\gamma_3)\\ &\quad\cdot f_{\mathcal{G}}(\gamma_1)f_{\mathcal{G}}(\gamma_2)f_{\mathcal{G}}(\gamma_3)\,d\gamma_1\,d\gamma_2\,d\gamma_3\\ &= \mu^s_{2\gamma}\left[(\mu^c_\gamma)^2-(\mu^s_\gamma)^2\right]-2\mu^c_{2\gamma}\mu^c_\gamma\mu^s_\gamma=\overline{\delta}^{\,s}_\gamma \end{aligned}\tag{139}$$

$$\boxed{\lfloor\tfrac{j-1}{L}\rfloor=\lfloor\tfrac{l-1}{L}\rfloor\neq\lfloor\tfrac{i-1}{L}\rfloor\neq\lfloor\tfrac{k-1}{L}\rfloor}\;:\gamma_{ijkl}=-2\gamma_1+\gamma_2+\gamma_3$$

$$\delta^c_{\gamma,\,ijkl}=\overline{\delta}^{\,c}_\gamma,\qquad \delta^s_{\gamma,\,ijkl}=-\overline{\delta}^{\,s}_\gamma \tag{140}$$

$$\boxed{\text{all other cases}}\;:\;\gamma_{ijkl}=\pm(\gamma_1-\gamma_2)$$

$$\delta^c_{\gamma,\,ijkl}=\sigma^c_\gamma,\qquad \delta^s_{\gamma,\,ijkl}=0. \tag{141}$$

- **Four different values:** $a_{ijkl} = a_1 a_2 a_3 a_4$, $\gamma_{ijkl} = \gamma_1 - \gamma_2 + \gamma_3 - \gamma_4$

$$\begin{aligned}\delta^a_{ijkl} &= \int_{a_1}\int_{a_2}\int_{a_3}\int_{a_4} a_1a_2a_3a_4\\ &\quad\cdot f_\alpha(a_1)f_\alpha(a_2)f_\alpha(a_3)f_\alpha(a_4)\,da_1\,da_2\,da_3\,da_4\\ &=\mu_a^4 \end{aligned}\tag{142}$$

$$\begin{aligned}\delta^c_{\gamma,\,ijkl} &= \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3}\int_{\gamma_4}\cos(\gamma_1-\gamma_2+\gamma_3-\gamma_4)\\ &\quad\cdot f_{\mathcal{G}}(\gamma_1)f_{\mathcal{G}}(\gamma_2)f_{\mathcal{G}}(\gamma_3)f_{\mathcal{G}}(\gamma_4)\,d\gamma_1\,d\gamma_2\,d\gamma_3\,d\gamma_4\\ &= \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3}\int_{\gamma_4}[\cos(\gamma_1-\gamma_2)\cos(\gamma_3-\gamma_4)\\ &\quad-\sin(\gamma_1-\gamma_2)\sin(\gamma_3-\gamma_4)]\\ &\quad\cdot f_{\mathcal{G}}(\gamma_1)f_{\mathcal{G}}(\gamma_2)f_{\mathcal{G}}(\gamma_3)f_{\mathcal{G}}(\gamma_4)\,d\gamma_1\,d\gamma_2\,d\gamma_3\,d\gamma_4\\ &=(\sigma^c_\gamma)^2 \end{aligned}\tag{143}$$

$$\begin{aligned}\delta^s_{\gamma,\,ijkl} &= \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3}\int_{\gamma_4}\sin(\gamma_1-\gamma_2+\gamma_3-\gamma_4)\\ &\quad\cdot f_{\mathcal{G}}(\gamma_1)f_{\mathcal{G}}(\gamma_2)f_{\mathcal{G}}(\gamma_3)f_{\mathcal{G}}(\gamma_4)\,d\gamma_1\,d\gamma_2\,d\gamma_3\,d\gamma_4\\ &= \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3}\int_{\gamma_4}[\sin(\gamma_1-\gamma_2)\cos(\gamma_3-\gamma_4)\\ &\quad+\cos(\gamma_1-\gamma_2)\sin(\gamma_3-\gamma_4)]\\ &\quad\cdot f_{\mathcal{G}}(\gamma_1)f_{\mathcal{G}}(\gamma_2)f_{\mathcal{G}}(\gamma_3)f_{\mathcal{G}}(\gamma_4)\,d\gamma_1\,d\gamma_2\,d\gamma_3\,d\gamma_4\\ &=0. \end{aligned}\tag{144}$$

For a *symmetric phase pdf* $f_{\mathcal{G}}(\gamma)$, i.e., a function for which $f_{\mathcal{G}}(\gamma_c+\gamma)=f_{\mathcal{G}}(\gamma_c-\gamma)$, $\forall\gamma$, for a certain γ_c, it can easily be proved that $\overline{\delta}^{\,s}_\gamma=0$ since

$$\begin{aligned}\overline{\delta}^{\,s}_\gamma &= \int\int\int_{\gamma_c-\gamma_I}^{\gamma_c+\gamma_I}\sin(2\gamma_1-\gamma_2-\gamma_3)\\ &\quad\cdot f_{\mathcal{G}}(\gamma_1)f_{\mathcal{G}}(\gamma_2)f_{\mathcal{G}}(\gamma_3)\,d\gamma_1\,d\gamma_2\,d\gamma_3 &(145)\\ &= \int\int\int_{-\gamma_I}^{0}\sin(2\gamma_1-\gamma_2-\gamma_3)f_{\mathcal{G}}(\gamma_c+\gamma_1)\\ &\quad\cdot f_{\mathcal{G}}(\gamma_c+\gamma_2)f_{\mathcal{G}}(\gamma_c+\gamma_3)\,d\gamma_1\,d\gamma_2\,d\gamma_3\\ &\quad+\int\int\int_{0}^{\gamma_I}\sin(2\gamma_1-\gamma_2-\gamma_3)f_{\mathcal{G}}(\gamma_c+\gamma_1)\\ &\quad\cdot f_{\mathcal{G}}(\gamma_c+\gamma_2)f_{\mathcal{G}}(\gamma_c+\gamma_3)\,d\gamma_1\,d\gamma_2\,d\gamma_3 &(146)\\ &= \int\int\int_{\gamma_I}^{0}-\sin(2\gamma_1-\gamma_2-\gamma_3)f_{\mathcal{G}}(\gamma_c-\gamma_1)\\ &\quad\cdot f_{\mathcal{G}}(\gamma_c-\gamma_2)f_{\mathcal{G}}(\gamma_c-\gamma_3)(-d\gamma_1)(-d\gamma_2)(-d\gamma_3)\\ &\quad+\int\int\int_{0}^{\gamma_I}\sin(2\gamma_1-\gamma_2-\gamma_3)f_{\mathcal{G}}(\gamma_c+\gamma_1)\\ &\quad\cdot f_{\mathcal{G}}(\gamma_c+\gamma_2)f_{\mathcal{G}}(\gamma_c+\gamma_3)\,d\gamma_1\,d\gamma_2\,d\gamma_3\\ &=0, &(147)\end{aligned}$$

such that for $\gamma_I=\infty$ we obtain

$$\begin{aligned}\overline{\delta}^{\,s}_\gamma &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin(2\gamma_1-\gamma_2-\gamma_3)\\ &\quad\cdot f_{\mathcal{G}}(\gamma_1)f_{\mathcal{G}}(\gamma_2)f_{\mathcal{G}}(\gamma_3)\,d\gamma_1\,d\gamma_2\,d\gamma_3=0. \end{aligned}\tag{148}$$

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