Asymptotics of Robin eigenvalues on sharp infinite cones

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Abstract

Let $\omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. For $\varepsilon > 0$ and $n \in \mathbb{N}$ consider the infinite cone

$$\Omega_\varepsilon := \left\{(x_1, x') \in (0, \infty) \times \mathbb{R}^n : x' \in \varepsilon x_1 \omega\right\} \subset \mathbb{R}^{n+1}$$

and the operator Q_ε^α acting as the Laplacian $u \mapsto -\Delta u$ on Ω_ε with the Robin boundary condition $\partial_\nu u = \alpha u$ at $\partial\Omega_\varepsilon$, where ∂_ν is the outward normal derivative and $\alpha > 0$. We look at the dependence of the eigenvalues of Q_ε^α on the parameter ε: this problem was previously addressed for $n = 1$ only (in that case, the only admissible ω are finite intervals). In the present work we consider arbitrary dimensions $n \geq 2$ and arbitrarily shaped "cross-sections" ω and look at the spectral asymptotics as ε becomes small, i.e. as the cone becomes "sharp" and collapses to a half-line. It turns out that the main term of the asymptotics of individual eigenvalues is determined by the single geometric quantity

$$N_\omega := \frac{\mathrm{Vol}_{n-1}\partial\omega}{\mathrm{Vol}_n \omega}.$$

More precisely, for any fixed $j \in \mathbb{N}$ and $\alpha > 0$ the jth eigenvalue $E_j(Q_\varepsilon^\alpha)$ of Q_ε^α exists for all sufficiently small $\varepsilon > 0$ and satisfies

$$E_j(Q_\varepsilon^\alpha) = -\frac{N_\omega^2\, \alpha^2}{(2j + n - 2)^2\, \varepsilon^2} + O\left(\frac{1}{\varepsilon}\right) \text{ as } \varepsilon \to 0^+.$$

The paper also covers some aspects of Sobolev spaces on infinite cones, which can be of independent interest.

Keywords: Laplacian, Robin boundary condition, asymptotics of eigenvalues, spectral problems in unbounded domains

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1 Introduction

Let $\omega \subset \mathbb{R}^n$ be a bounded domain (connected open set) with Lipschitz boundary. For $\varepsilon > 0$ consider the open set

$$\Omega_\varepsilon := \Big\{(x_1, x') \in (0,\infty) \times \mathbb{R}^n : x' \in \varepsilon x_1 \omega\Big\} \subset \mathbb{R}^{n+1}.$$

Geometrically, the set Ω_ε is an infinite cone in $\mathbb{R}^{n+1}$ such that the intersection of Ω_ε with the hyperplane $x_1 = a$ gives the set $\varepsilon a \omega$. We are interested in some spectral properties of a Robin Laplacian on Ω_ε as ε becomes small, i.e. when the cone becomes "sharp" and collapses to the half-line $(0,\infty) \times \{0\}$. Namely, for $\alpha > 0$ denote by Q_ε^α the self-adjoint operator in $L^2(\Omega_\varepsilon)$ generated by the closed, densely defined, symmetric bilinear form

$$q_\varepsilon^\alpha(u,u) = \int_{\Omega_\varepsilon} |\nabla u|^2 \,\mathrm{d}x - \alpha \int_{\partial\Omega_\varepsilon} u^2 \,\mathrm{d}\sigma, \quad D(q_\varepsilon^\alpha) = H^1(\Omega_\varepsilon),$$

where $\mathrm{d}\sigma$ stands for the n-dimensional Hausdorff measure. The semiboundedness and the closedness are not completely obvious as Ω_ε is unbounded and may have a non-Lipschitz singularity at the origin: we discuss these aspects in detail in Subsection 2.5 below. Informally, the operator Q_ε^α can be viewed as the positive Laplacian, $u \mapsto -\Delta u$, with the Robin boundary condition $\partial_\nu u = \alpha u$, where ∂_ν is the outward normal derivative; we refer to [3, 9, 10] for a discussion of various aspects related to the precise description of the operator domain. Such operators are often referred to as Robin Laplacians with negative parameters [6] due to the negative contribution of the boundary term in the bilinear form. The cone Ω_ε is invariant with respect to the dilations $x \mapsto tx$ for any $t > 0$, and standard arguments show the unitary equivalence $Q_\varepsilon^\alpha \simeq \alpha^2 Q_\varepsilon^1$. Hence, it will be convenient to consider $\alpha = 1$ only and to study the operator and the form

$$Q_\varepsilon := Q_\varepsilon^1, \quad q_\varepsilon := q_\varepsilon^1.$$

For a review of spectral problems with Robin boundary conditions we refer to [6]. In particular, the eigenvalues of Robin Laplacians on infinite cones play a central role in the strong coupling asymptotics of Robin eigenvalues on general domains. Namely, if Ω is an open set of some large class and $T^{\Omega,\alpha}$ is the Robin Laplacian on Ω defined as the operator associated with the symmetric bilinear form

$$t^{\Omega,\alpha}(u,u) = \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \alpha \int_{\partial\Omega} u^2 \,\mathrm{d}\sigma, \quad u \in H^1(\Omega),$$

then the lower edge $\Lambda_1(T^{\Omega,\alpha})$ of the spectrum of $T^{\Omega,\alpha}$ satisfies

$$\Lambda_1(T^{\Omega,\alpha}) = \alpha^2 \inf_{x\in\partial\Omega} \Lambda_1(T^{U_x,1}) + o(\alpha^2) \quad \text{as } \alpha \to +\infty,$$

where $T^{U_x,1}$ is the Robin Laplacian on the infinite tangent cone U_x at $x \in \partial\Omega$. We refer to [5, 24] for technical details and precise definitions and to [13–16, 19, 28] for a more precise eigenvalue analysis under more specific regularity assumptions. The function $\alpha \mapsto \Lambda_1(T^{\Omega,\alpha})$ plays a role in the study of some non-linear equations as discussed in [23]. Eigenvalues and eigenfunctions of sharp cones can be used to

produce counterexamples to spectral gap estimates [21]. In addition, such operators attract some attention as examples of geometric "long-range" configurations producing an infinite discrete spectrum [4, 8, 27]. Let us summarize the available spectral information for Q_ε.

The essential spectrum of Q_ε depends in a non-trivial way on ω and ε. If ω has smooth boundary, then in virtue of [27, Thm. 1] the essential spectrum of Q_ε is $[-1, +\infty)$, as Ω_ε is smooth outside the origin. For non-smooth ω the essential spectrum is determined through an iterative procedure and can look differently: see the detailed discussion in [5].

If ω is the unit ball centered at the origin of $\mathbb{R}^n$, then Q_ε is a round cone whose lateral surface forms the constant angle $\theta := \arctan \varepsilon$ with the central axis, and the bottom of the spectrum of Q_ε is the eigenvalue

$$E_1(Q_\varepsilon) = -\frac{1}{\sin^2\theta} \equiv -\frac{1+\varepsilon^2}{\varepsilon^2} \tag{1}$$

with eigenfunction $\psi(x_1, x') = \exp(-x_1/\sin\theta)$. In fact, only $n = 1$ and $n = 2$ were considered explicitly, see e.g. [24, Lem. 2.6] and [18, Prop. 4.2], but the constructions literally hold for arbitrary dimensions n.

The case $n = 1$ (Ω_ε is an infinite planar sector) was studied in detail in [20]. The only admissible sets ω are finite intervals, so without loss of generality take $\omega := (-1, 1)$. In [20] it was shown that the discrete spectrum of Q_ε is always finite, but the number of eigenvalues grows unboundedly as ε becomes small, and for each fixed $j \in \mathbb{N}$ (we use the convention $0 \notin \mathbb{N}$) the jth eigenvalue $E_j(Q_\varepsilon)$ behaves as

$$E_j(Q_\varepsilon) = -\frac{1}{(2j-1)^2\varepsilon^2} + O(1) \text{ as } \varepsilon \to 0^+. \tag{2}$$

Some explicit formulas for eigenpairs of Q_ε in this particular case were obtained in [25], but it is unclear if the constructed family exhausts the whole discrete spectrum.

If $n \geq 2$, the discrete spectrum of Q_ε may be infinite. For example, if $n = 2$ and ω is simply connected with smooth boundary, then the infiniteness of the discrete spectrum follows by [27, Cor. 8], as the complement of Ω_ε is not a convex set (similar arguments apply in higher dimensions: we refer to [27] for details). On the other hand, for polyhedral ω the discrete spectrum can be finite. For example, if one chooses ω in such a way that Ω_1 is an isometric copy of $(0, \infty)^{n+1}$, then an easy analysis based on the separation of variables method shows that the discrete spectrum of Q_1 consists of a single eigenvalue $-(n+1)$. For $n = 2$ and smooth ω, the accumulation rate of eigenvalues at the bottom of essential spectrum was studied in [4]. Furthermore, in [18] it was shown that round infinite cones maximize the first eigenvalue among all cones with the same perimeter of the spherical cross-section. Various two-sided estimates for the bottom of the spectrum were obtained in [24]. In particular, it was shown that the lowest eigenvalue can be computed explicitly if the spherical cross-section of Ω_ε is a spherical polygon admitting an inscribed circle.

In the present work we complement the above results by computing the asymptotics of *individual* eigenvalues of Q_ε for small ε in arbitrary dimensions and arbitrary cross-sections ω. It turns out that the main term in the asymptotics depends on a single geometric constant N_ω given in (3) and, hence, it is rather insensitive to the regularity of ω. Our result reads as follows:

Theorem 1. *Let $j \in \mathbb{N}$, then Q_ε has at least j discrete eigenvalues below the bottom of the essential spectrum for all sufficiently small $\varepsilon > 0$, and its jth eigenvalue $E_j(Q_\varepsilon)$ satisfies*

$$E_j(Q_\varepsilon) = -\frac{N_\omega^2}{(2j+n-2)^2\,\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right) \ \text{as } \varepsilon \to 0^+, \quad N_\omega := \frac{\mathrm{Vol}_{n-1}\,\partial\omega}{\mathrm{Vol}_n\,\omega}. \tag{3}$$

For $n = 1$ and $\omega = (-1, 1)$ one has $N_\omega = 1$, and the result follows directly from (2), and all other intervals ω are easily included by applying suitable reparametrizations. Hence, *for the rest of the text we explicitly assume $n \geq 2$*. Remark that if ω is a unit ball centered at the origin, then one has $N_\omega = n$, and the exact formula (1) has the form (3) with $j = 1$ and a more accurate remainder estimate. Based on these observations one may expect that the remainder estimate in (3) is not optimal. We further remark that if the volume $\mathrm{Vol}_n\,\omega$ or the surface area $\mathrm{Vol}_{n-1}\,\partial\omega$ is fixed, then the quantity N_ω is minimized by the ball due to the classical isoperimetric inequality. Hence, the sharp cones Ω_ε whose cross-section ω are balls maximize the main term in (3) among all sharp cones with cross-sections of the same volume or surface area.

Our proof is variational and based on the min-max principle, and its main ingredient is a kind of asymptotic separation of the variables x_1 and x', which is quite similar to [20], but the analysis in the x'-direction is much more involved and uses some coordinate transforms similar to [22]. Various proof steps are explained in greater detail in Subsection 2.6 below. We remark that in Subsection 2.2 and 2.5 we prove some results on Sobolev spaces on Ω_ε (which is unbounded and may be non-Lipschitz) that are needed for the spectral analysis: this part of the text may be of its own interest.

2 Preparations for the proof

2.1 Min-max principle

If T is a self-adjoint operator on an infinite-dimensional Hilbert space $\mathcal{H}$, we denote by $E_j(T)$ its jth eigenvalue (when enumerated in the non-decreasing order and counted according to the multiplicities), if it exists. All operators we consider are real (i.e. map real-valued functions to real-valued functions), and we prefer to work with real Hilbert spaces in order to have shorter expressions. The spectrum and the essential spectrum of T will be denoted by $\operatorname{spec} T$ and $\operatorname{spec}_{\mathrm{ess}} T$ respectively.

Let t be the bilinear form for T, with domain $D(t)$, and let $D \subset D(t)$ be any dense subset (with respect to the scalar product induced by t). Consider the following "variational eigenvalues"

$$\Lambda_j(T) := \inf_{\substack{V \subset D \\ \dim V = j}} \sup_{\substack{u \in V \\ u \neq 0}} \frac{t(u,u)}{\langle u, u\rangle_{\mathcal{H}}},$$

which are independent of the choice of D. One easily sees that $j \mapsto \Lambda_j(T)$ is non-decreasing. Furthermore, if one denotes $\Sigma := \inf \operatorname{spec}_{\mathrm{ess}} T$ for $\operatorname{spec}_{\mathrm{ess}} T \neq \emptyset$ and $\Sigma := +\infty$ otherwise, then it is known [30, Section XIII.1] that only two cases are possible:

- For all $j \in \mathbb{N}$ there holds $\Lambda_j(T) < \Sigma$. Then the spectrum of T in $(-\infty, \Sigma)$ consists of infinitely many discrete eigenvalues $E_j(T) \equiv \Lambda_j(T)$ with $j \in \mathbb{N}$.
- For some $N \in \mathbb{N} \cup \{0\}$ there holds $\Lambda_{N+1}(T) \geq \Sigma$, while $\Lambda_j(T) < \Sigma$ for all $j \leq N$. Then T has exactly N discrete eigenvalues in $(-\infty, \Sigma)$ and $E_j(T) = \Lambda_j(T)$ for $j \in \{1, \dots, N\}$, while $\Lambda_j(T) = \Sigma$ for all $j \geq N+1$.

In all cases there holds $\lim_{j\to\infty} \Lambda_j(T) = \Sigma$, and if for some $j \in \mathbb{N}$ one has $\Lambda_j(T) < \Sigma$, then $E_j(T) = \Lambda_j(T)$. In particular, if for some $j \in \mathbb{N}$ one has the strict inequality $\Lambda_j(T) < \Lambda_{j+1}(T)$, then $E_j(T) = \Lambda_j(T)$.

2.2 Density in Sobolev spaces on cones

We prefer to discuss in detail some properties of Sobolev spaces on Ω_ε. In fact, particular attention should be paid to such aspects, as Ω_ε is unbounded and, in general, not with Lipschitz boundary, and it does not satisfy the standard assumptions for trace theorems and other important assertions discussed in most books.

By $\mathrm{d}\sigma$ and $\mathrm{d}\tau$ we will denote the integration with respect to the n- and $(n-1)$-dimensional Hausdorff measures, respectively.

For an open set $\Omega \subset \mathbb{R}^m$ and $k \in \mathbb{N}$ the k-th Sobolev space $H^k(\Omega)$ is defined as

$$H^k(\Omega) := \Big\{ u \in L^2(\Omega) : \partial^\alpha u \in L^2(\Omega) \text{ for all } |\alpha| \leq k \}$$

with all derivatives taken in the sense of distributions, and it is a Hilbert space with respect to the scalar product

$$\langle u, v \rangle_{H^k(\Omega)} := \sum_{|\alpha| \leq k} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2(\Omega)}.$$

By $C^\infty(\overline{\Omega})$ one denotes the set of functions defined on Ω that can be extended to functions in $C_c^\infty(\mathbb{R}^m)$.

One says that an open set $\Omega \subset \mathbb{R}^m$ has C^k (respectively Lipschitz) boundary, if for any $p \in \partial\Omega$ there exist Cartesian coordinates $(y_1, \dots, y_m)$ centered at p, a C^k (respectively Lipschitz) function h of $m-1$ variables, defined on an open neighborhood of 0 in $\mathbb{R}^{m-1}$ and with $h(0, \dots, 0) = 0$, and $\varepsilon > 0$ such that

$$\Omega \cap B_\varepsilon(p) = \Big\{ y = (y_1, \dots, y_m) \in B_\varepsilon(0) : \, y_m < h(y_1, \dots, y_{m-1}) \Big\}.$$

Most assertions used in the theory of Sobolev spaces (some density and extension results, trace theorems) are usually formulated for bounded open sets with Lipschitz boundaries. On the other hand, the cone Ω_ε has in general *not* even a C^0 boundary: for example, if $n = 2$ and ω is an annulus, $\omega = \{(x_1, x_2) : 1 < x_1^2 + x_2^2 < 4\}$, then one easily sees that Ω_ε cannot be represented as one of the sides of the graph of a continuous function near the vertex 0. Moreover, further common assumptions used in the theory of Sobolev spaces (e.g. the segment condition or the cone condition) fail as well.

We collect some known facts on $H^k(\Omega)$ in the following proposition:

Proposition 2. *(A) The space*

$$H^1_\infty(\Omega) := \Big\{u \in H^1(\Omega) : u \in C^\infty(\Omega) \cap L^\infty(\Omega),\ \operatorname{supp} u \text{ is bounded}\Big\}$$

is dense in $H^1(\Omega)$. (Remark that there are no additional assumptions on Ω.)

(B) If Ω has C^0 boundary, then $C^\infty(\overline{\Omega})$ is dense in $H^k(\Omega)$ for any $k \in \mathbb{N}$.

(C) If Ω is bounded and has Lipschitz boundary, then:

(C.1) for any $k \in \mathbb{N}$, any function in $H^k(\Omega)$ can be extended to a function in $H^k(\mathbb{R}^m)$.

(C.2) the linear map $C^\infty(\overline{\Omega}) \ni u \mapsto u|_{\partial\Omega} \in L^2(\partial\Omega)$ uniquely extends by continuity to a bounded linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$. Moreover, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\int_{\partial\Omega} (\gamma_0 u)^2 d\sigma_{m-1} \le \varepsilon \int_\Omega |\nabla u|^2 \,\mathrm{d}x + C_\varepsilon \int_\Omega u^2 \,\mathrm{d}x$$

for all $u \in H^1(\Omega)$, where σ_{m-1} is the $(m-1)$-dimensional Hausdorff measure.

We refer to [26, Theorem in Sec. 1.4.3] for (A), to [26, Theorem 1 in Sec. 1.4.2] for (B), to [1, Thm. 5.2.4] for (C.1) and to [12, Theorem 1.5.1.10] for (C.2). Remark that one usually writes simply u instead of $\gamma_0 u$ in the integrals over the boundary.

Now we pass to the discussion of Sobolev spaces on the infinite cones Ω_ε. We start with several preparation steps.

Lemma 3. *Let $-\infty < a < b < \infty$, then the cylinder $\Omega := (a,b) \times \omega \subset \mathbb{R}^{n+1}$ has Lipschitz boundary.*

Proof. Let $p \in \partial\Omega$, then the following cases are possible.

Case 1: $p = (a', p')$ with $a' \in (a,b)$ and $p' \in \partial\omega$. As ω has Lipschitz boundary, there exists Cartesian coordinates $(y_1, \dots, y_n)$ in $\mathbb{R}^n$ centered at p' and a Lipschitz function h with $h(0) = 0$ such that ω coincides with $\{y : y_n < h(y_1, \dots, y_{n-1})\}$ near p'. Denote $z := x_1 - a'$, then $(z, y_1, \dots, y_n)$ are Cartesian coordinates in $\mathbb{R}^{n+1}$ centered at p, and Ω near p coincides with $\{(z,y) : y_n < H(z, y_1, \dots, y_{n-1})\}$ for the function $H(z, y_1, \dots, y_{n-1}) := h(y_1, \dots, y_{n-1})$, which is obviously Lipschitz.

Case 2a: $p = (a, x')$ with $x' \in \omega$, then Ω near p coincides with $\{(z,y) : z < 0\}$, where $y = (y_1, \dots, y_n)$ are arbitrary Cartesian coordinates in $\mathbb{R}^n$ centered at x' and $z := a - x_1$: remark that $(z, y_1, \dots, y_n)$ are Cartesian coordinates in $\mathbb{R}^{n+1}$ centered at p, and the zero function is obviously Lipschitz. Case 2b: $p = (b, x')$ with $x' \in \omega$ is treated analogously.

Case 3a: $p = (a, p')$ with $p' \in \partial\omega$ (the most difficult one). As ω has Lipschitz boundary, there exist Cartesian coordinates $(y_1, \dots, y_n)$ in $\mathbb{R}^n$ centered at p' and a Lipschitz function h with $h(0) = 0$ such that ω coincides with $\{y : y_n < h(y_1, \dots, y_{n-1})\}$ near p'. Remark that Ω near p is then determined by the two inequalities

$$x_1 > a, \quad y_n < h(y_1, \dots, y_{n-1}). \tag{4}$$

In order to bring these conditions into the required form we pick $\theta \in (0, \frac{\pi}{2})$ and apply a rotation by the angle θ around p in the (x_1, y_n)-plane. Namely, consider the Cartesian coordinates $(z, y_1, \dots, y_{n-1}, w)$ with the previous $y_1, \dots, y_{n-1}$ and

$$\begin{pmatrix} x_1 - a \\ y_n \end{pmatrix} = z \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + w \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Clearly, the new coordinates are centered at p, and the above inequalities (4) determining Ω near p take the form

$$w < \frac{\cos\theta}{\sin\theta}, \quad w < -\frac{\sin\theta}{\cos\theta} z + \frac{1}{\cos\theta} h(y_1, \dots, y_{n-1}), \tag{5}$$

which can be rewritten as

$$w < H(z, y_1, \dots, y_{n-1}) := \min\left\{\frac{\cos\theta}{\sin\theta} z, -\frac{\sin\theta}{\cos\theta} z + \frac{1}{\cos\theta} h(y_1, \dots, y_{n-1})\right\}.$$

The function H is Lipschitz (as it is the minimum of two Lipschitz functions), hence, one has a required representation of Ω near p. The case 3b: $p = (b, p')$ with $p' \in \partial\omega$ is considered analogously. □

Lemma 4. *Let $-\infty < a < b < \infty$ and $\Omega := (a, b) \times \omega \subset \mathbb{R}^{n+1}$. Let $[c, c'] \subset (a, b)$ and $u \in H^1(\Omega)$ such that $u(x_1, x') = 0$ for $x_1 \notin [c, c']$. Let $0 < \delta < \min\{c - a, b - c'\}$, then for any $\varepsilon > 0$ there exists $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ such that $\|u - \varphi\|_{H^1(\Omega)} < \varepsilon$ and $\varphi(x_1, x') = 0$ for all $x_1 \notin [c - \delta, c' + \delta]$.*

Proof. By Lemma 3 and Proposition 2(C.1) the function u can be extended to a function $v' \in H^1(\mathbb{R}^{n+1})$. Choose $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi(s) = 1$ for $s \in [c, c']$ and $\operatorname{supp}\chi \in \left[c - \frac{\delta}{2}, c' + \frac{\delta}{2}\right]$, and, in addition, choose $\chi_0 \in C_c^\infty(\mathbb{R}^n)$ with $\chi_0 = 1$ on ω. Then the function $v : (x_1, x') \mapsto \chi(x_1)\chi_0(x')v'(x_1, x')$ belongs to $H^1(\mathbb{R}^{n+1})$, is an extension of u, has compact support, and $v(x_1, x') = 0$ for all $x_1 \notin \left[c - \frac{\delta}{2}, c' + \frac{\delta}{2}\right]$.

Let $\rho \in C_c^\infty(\mathbb{R}^{n+1})$ with

$$\rho(y) = 0 \text{ for } |y| \geq 1, \quad \int_{\mathbb{R}^{n+1}} \rho(y)\,\mathrm{d}y = 1,$$

and for $t > 0$ consider the functions $\rho_t : x \mapsto t^{-(n+1)}\rho(t^{-1}x)$. Then $v_t := v * \rho_t \in C_c^\infty(\mathbb{R}^{n+1})$, where $*$ denotes the convolution product, and $\|v_t - v\|_{H^1(\mathbb{R}^{n+1})} \to 0$ for $t \to 0^+$. Hence, there exists some $t_0 > 0$ such that $\|v_t - v\|_{H^1(\mathbb{R}^{n+1})} < \varepsilon$ for all $t \in (0, t_0)$.

Furthermore, the definition of the convolution product implies the inclusion $\operatorname{supp} v_t \subset \operatorname{supp} v + \overline{B}_t(0)$. In particular, if $t < \frac{\delta}{2}$, then one has $v_t(x_1, x') = 0$ for all $x_1 \notin [c - \delta, c' + \delta]$. Now pick any $0 < t < \min\left\{t_0, \frac{\delta}{2}\right\}$ and denote $\varphi := v_t$, then

$$\|\varphi - u\|_{H^1(\Omega)} = \|v_t - v\|_{H^1(\Omega)} \leq \|v_t - v\|_{H^1(\mathbb{R}^{n+1})} < \varepsilon,$$

so φ has all the required properties. □

For an open interval $I \subset (0, \infty)$ we denote

$$C_I^\infty(\overline{\Omega}_\varepsilon) := \left\{u \in C^\infty(\overline{\Omega}_\varepsilon) : \exists\, [b, c] \subset I \text{ such that } u(x) = 0 \text{ for } x_1 \notin [b, c]\right\}. \tag{6}$$

Proposition 5. *The subspace $C^\infty_{(0,\infty)}(\overline{\Omega}_\varepsilon)$ is dense in $H^1(\Omega_\varepsilon)$.*

Proof. First remark that $H^1_\infty(\Omega_\varepsilon)$ is dense in $H^1(\Omega_\varepsilon)$ by Proposition 2(A). So we need to show that any function from $H^1_\infty(\Omega_\varepsilon)$ can be approximated by functions from $C^\infty_{(0,\infty)}(\overline{\Omega}_\varepsilon)$ in the H^1-norm.

Let $v \in H^1_\infty(\Omega_\varepsilon)$, then there exists some $c \in (0,\infty)$ such that $v(x_1,x') = 0$ for $x_1 > c$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a C^∞-function with $0 \le \chi \le 1$, $\chi(s) = 0$ for $s < \frac{1}{2}$, and $\chi(s) = 1$ for $s > 1$. For $\delta > 0$ consider the functions

$$v_\delta : (x_1,x') \mapsto \chi\Big(\frac{x_1}{\delta}\Big) v(x_1,x').$$

We have

$$\|v_\delta - v\|^2_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \Big|1 - \chi\Big(\frac{x_1}{\delta}\Big)^2\Big|\, v(x_1,x')^2 \,\mathrm{d}x \le \int_{\Omega_\varepsilon \cap \{x_1<\delta\}} v(x_1,x')^2 \,\mathrm{d}x \overset{\delta\to 0^+}{\longrightarrow} 0.$$

Furthermore,

$$\begin{aligned}
\partial_1 v_\delta(x_1,x') &= \frac{1}{\delta}\chi'\Big(\frac{x_1}{\delta}\Big) v(x_1,x') + \chi\Big(\frac{x_1}{\delta}\Big)\partial_1 v(x_1,x'),\\
\partial_j v_\delta(x_1,x') &= \chi\Big(\frac{x_1}{\delta}\Big)\partial_j v(x_1,x') \text{ for } j \ge 2.
\end{aligned}$$

For every $j \ge 2$ one obtains

$$\begin{aligned}
\|\partial_j v_\delta - \partial_j v\|^2_{L^2(\Omega_\varepsilon)} &= \int_\Omega \Big|1 - \chi\Big(\frac{x_1}{\delta}\Big)^2\Big| \Big|\partial_j v(x_1,x')\Big|^2 \,\mathrm{d}x\\
&\le \int_{\Omega_\varepsilon \cap \{x_1<\delta\}} \Big|\partial_j v(x_1,x')\Big|^2 \,\mathrm{d}x \overset{\delta\to 0^+}{\longrightarrow} 0.
\end{aligned}$$

In addition, using $(x+y)^2 \le 2(x^2+y^2)$ we estimate

$$\begin{aligned}
\|\partial_1 v_\delta - \partial_1 v\|^2_{L^2(\Omega_\varepsilon)} &\le 2\int_{\Omega_\varepsilon} \Big|1 - \chi\Big(\frac{x_1}{\delta}\Big)^2\Big| \Big|\partial_1 v(x_1,x')\Big|^2 \,\mathrm{d}x\\
&\quad + \frac{2}{\delta^2}\int_{\Omega_\varepsilon} \chi'\Big(\frac{x_1}{\delta}\Big)^2 v(x_1,x')^2 \,\mathrm{d}x \qquad (7)\\
&\le \int_{\Omega_\varepsilon \cap \{x_1<\delta\}} \Big|\partial_1 v(x_1,x')\Big|^2 \,\mathrm{d}x + \frac{2}{\delta^2}\,\|\chi'\|^2_\infty \|v\|^2_\infty \int_{\Omega_\varepsilon \cap \{x_1<\delta\}} \mathrm{d}x.
\end{aligned}$$

The first summand on the right-hand converges to 0 as $\delta \to 0^+$ as $\partial_1 v \in L^2(\Omega_\varepsilon)$. We further note that

$$\int_{\Omega_\varepsilon \cap \{x_1<\delta\}} \mathrm{d}x = \int_0^\delta \int_{\varepsilon x_1 \omega} \mathrm{d}x'\,\mathrm{d}x_1 = \varepsilon^n \operatorname{Vol}_n \omega \int_0^\delta x_1^n \,\mathrm{d}x_1 = \varepsilon^n \operatorname{Vol}_n \omega\, \frac{\delta^{n+1}}{n+1},$$

and the second summand on the right-hand side of (7) is estimated from above by $\frac{2}{n+1}\,\varepsilon^n \|\chi'\|^2_\infty \|v\|^2_\infty \operatorname{Vol}_n \omega \delta^{n-1}$, which converges to 0 for $\delta \to 0^+$ due to $n \ge 2$. We have proved that v_δ converges to v in $H^1(\Omega_\varepsilon)$ as $\delta \to 0^+$. Remark that $v_\delta(x_1,x') = 0$ for $x_1 \notin [\delta/2, c]$, therefore, the above constructions show that the subspace

$$\begin{aligned}
\mathcal{D} := \Big\{ u \in H^1(\Omega_\varepsilon) \cap C^\infty(\Omega_\varepsilon) \cap L^\infty(\Omega_\varepsilon) :\\
\exists\, [b,c] \subset (0,\infty) \text{ such that } u(x_1,x') = 0 \text{ for } x_1 \notin [b,c] \Big\},
\end{aligned}$$

is dense in $H^1(\Omega_\varepsilon)$. Now it remains to check that each function for $\mathcal{D}$ can be approximated by functions from $C^\infty_{(0,\infty)}(\overline{\Omega}_\varepsilon)$ in $H^1(\Omega_\varepsilon)$.

Let $u \in \mathcal{D}$ and $[b,c] \subset (0,\infty)$ such that $u(x_1,x') = 0$ for $x_1 \notin [b,c]$. The map $X : (0,\infty)\times\mathbb{R}^n \ni (s,t) \mapsto (s,\varepsilon st) \in (0,\infty)\times\mathbb{R}^n$ is a diffeomorphism with $X\Big((0,\infty)\times\omega\Big) = \Omega_\varepsilon$. Pick an arbitrary $\delta \in (0,\frac{b}{2})$ and denote $\Omega' := (b-2\delta, c+2\delta)\times\omega$, then the function $u_X := u\circ X$ belongs to $H^1(\Omega')$.

Let $\mu > 0$, then by Lemma 4 one can find $\varphi^\mu_X \in C^\infty_c(\mathbb{R}^{n+1})$ with

$$\|u_X - \varphi^\mu_X\|_{H^1(\Omega')} < \mu, \quad \varphi^\mu_X(x_1,x') = 0 \text{ for all } x_1 \notin [b-\delta, c+\delta].$$

Then the functions

$$\varphi^\mu : \mathbb{R}^{n+1} \ni x \mapsto \begin{cases} \varphi^\mu_X\Big(X^{-1}(x)\Big), & x_1 > 0, \\ 0, & \text{otherwise} \end{cases}$$

belong to $C^\infty_c(\mathbb{R}^{n+1})$ and $\varphi^\mu(x_1,x') = 0$ for all $x_1 \notin [b-\delta, c+\delta]$, i.e. the restriction of φ^μ to Ω_ε belongs to $C^\infty_{(0,\infty)}(\overline{\Omega}_\varepsilon)$.

The supports of u and φ^μ are contained in $[b-\delta, c+\delta]\times\mathbb{R}^n$ and all derivatives of X and X^{-1} are uniformly bounded on the compact sets $\overline{\Omega'}$ and $\overline{\Omega}_\varepsilon \cap \{x_1 \in [b-2\delta, c+2\delta]\}$ respectively. Therefore, one can find some $C > 0$ such that

$$\|u - \varphi^\mu\|_{H^1(\Omega_\varepsilon)} \equiv \|u - \varphi^\mu\|_{H^1(\Omega_\varepsilon\cap\{x_1\in(b-2\delta,c+2\delta)\})} \le C\|u_X - \varphi^\mu_X\|_{H^1(\Omega')}$$

for all $\mu > 0$. As μ can be taken arbitrarily small, this concludes the proof. □

2.3 Robin Laplacian on ω

Given $r \in \mathbb{R}$ denote by B_r the self-adjoint operator in $L^2(\omega)$ generated by the closed symmetric bilinear form

$$b_r(f,f) = \int_\omega |\nabla f(t)|^2 \,\mathrm{d}t - r\int_{\partial\omega} f(t)^2 \,\mathrm{d}\tau(t), \qquad f \in H^1(\omega); \tag{8}$$

remark that b_r is semibounded from below due to Proposition 2. Informally, the operator B_r is the Laplacian $f \mapsto -\Delta f$ on ω with the Robin boundary condition $\partial_\nu f = rf$, with ∂_ν being the outward normal derivative. We will summarize some important spectral properties of B_r as follows.

Lemma 6. *The following assertions hold true:*

(a) For any $r \in \mathbb{R}$ the first eigenvalue $E_1(B_r)$ is simple, and the corresponding eigenfunction ψ_r can be chosen strictly positive with $\|\psi_r\|_{L^2(\omega)} = 1$.

(b) The mappings $\mathbb{R} \ni r \mapsto E_1(B_r) \in \mathbb{R}$ and $\mathbb{R} \ni r \mapsto \psi_r \in L^2(\omega)$ are C^∞.

(c) There exists $\varphi \in L^\infty(0,\infty)$ such that $E_1(B_r) = -N_\omega r + r^2\varphi(r)$ for all $r > 0$ and N_ω as defined in (3).

(d) Let $E^N_2 > 0$ be the second eigenvalue of the Neumann Laplacian on ω, then $\lim_{r\to 0} E_2(B_r) = E^N_2$.

(e) For any $r_0 > 0$ there exists $K > 0$ such that

$$\int_\omega \left|\partial_r \psi_r(y)\right|^2 \mathrm{d}y \le K \text{ for all } r \in (0, r_0). \tag{9}$$

Proof. Part (a) is proved for even more general Robin problems in [2, Sec. 4.2]. Both (b) and (d) follow from the fact that the operators B_r form a type (B) analytic family with respect to r, see [17, Ch. 7, §4], and (e) is a direct consequence of (b). To prove (c) we remark first that there exists $C > 0$ such that

$$-Cr^2 \le E_1(B_r) \le 0 \text{ as } r \to +\infty; \tag{10}$$

the lower bound is proved e.g. in [22, Corol. 2.2], and the upper bound follows from $b_r(1,1) < 0$ (which holds for all $r > 0$) by the min-max principle. Furthermore, by Eq. (4.16) in [6] one has

$$\frac{d}{dr} E_1(B_r)\Big|_{r=0} = -N_\omega,$$

and it follows that $E_1(B_r) = -N_\omega r + O(r^2)$ as $r \to 0^+$. By combining this asymptotics with (10) we arrive at the representation in (c). □

2.4 One-dimensional model operators

Given $\lambda > 0$ we consider the symmetric differential operator in $L^2(0,\infty)$ given by

$$C_c^\infty(0,\infty) \ni f \mapsto -f'' + \left(\frac{n^2 - 2n}{4s^2} - \frac{N_\omega}{\lambda s}\right) f \tag{11}$$

and denote by A_λ its Friedrichs extension. Remark that $n^2 - 2n \ge 0$ due to $n \ge 2$.

In [11, Chapter 8.3] the spectrum of A_λ was fully determined[1]: the essential spectrum is $[0, +\infty)$ and the negative eigenvalues are simple and are explicitly given by

$$E_j(A_\lambda) = \frac{E_j(A_1)}{\lambda^2} = -\frac{N_\omega^2}{(2j + n - 2)^2 \lambda^2}, \qquad j \in \mathbb{N}, \quad \lambda > 0. \tag{12}$$

In what follows we will need to work with truncated versions of A_λ. Namely, for $b > 0$ we denote by $M_{\lambda,b}$ and $\widetilde{M}_{\lambda,b}$ the Friedrichs extensions in $L^2(0,b)$ and $L^2(b,\infty)$ of the operators $C_c^\infty(0,b) \ni f \mapsto A_\lambda f$ and $C_c^\infty(b,\infty) \ni f \mapsto A_\lambda f$ respectively.

Remark that by construction the form domain of $M_{\lambda,b}$ is contained in $H_0^1(0,b)$, which implies that $M_{\lambda,b}$ has compact resolvent. We need to relate the eigenvalues of $M_{\lambda,b}$ to those of A_λ. As the bilinear form of A_λ extends that of $M_{\lambda,b}$, one has, due to the min-max principle,

$$E_j(M_{\lambda,b}) \ge E_j(A_\lambda) \text{ for any } b > 0,\ \lambda > 0,\ j \in \mathbb{N}. \tag{13}$$

Let us now obtain an asymptotic upper bound for $E_j(M_{\lambda,b})$.

Lemma 7. *Let $b > 0$ and $j \in \mathbb{N}$. Then there exist $K > 0$ and $\varepsilon_0 > 0$ such that*

$$E_j(M_{\varepsilon,b}) \le E_j(A_\varepsilon) + K \text{ for all } \varepsilon \in (0, \varepsilon_0).$$

[1] For $n = 2$ see p. 312 and for $n \ge 3$ see p. 294 in [11].

Proof. The proof is quite standard and uses a so-called IMS partition of unity [7, Sec. 3.1]. Let χ_1 and χ_2 be two smooth functions on $\mathbb{R}$ with $0 \le \chi_1, \chi_2 \le 1$, such that $\chi_1^2 + \chi_2^2 = 1$, $\chi_1(s) = 0$ for $s > \frac{3}{4}b$, $\chi_2(s) = 0$ for $s < \frac{1}{2}b$. We set $K := \|\chi_1'\|_\infty^2 + \|\chi_2'\|_\infty^2$. An easy computation shows that for any $f \in C_c^\infty(0,\infty)$ there holds

$$\begin{aligned}\int_0^\infty |f'|^2 \,\mathrm{d}s &= \int_0^\infty \left|(\chi_1 f)'\right|^2 \mathrm{d}s + \int_0^\infty \left|(\chi_2 f)'\right|^2 \mathrm{d}s - \int_0^\infty \Big(|\chi_1'|^2 + |\chi_2'|^2\Big) f^2 \,\mathrm{d}s \\ &\ge \int_0^\infty \left|(\chi_1 f)'\right|^2 \mathrm{d}s + \int_0^\infty \left|(\chi_2 f)'\right|^2 \mathrm{d}s - K\|f\|^2_{L^2(0,\infty)},\end{aligned}$$

which implies

$$\begin{aligned}\Big\langle f, A_\varepsilon f\Big\rangle_{L^2(0,\infty)} + K\|f\|^2_{L^2(0,\infty)} &\ge \Big\langle \chi_1 f, A_\varepsilon(\chi_1 f)\Big\rangle_{L^2(0,\infty)} + \Big\langle \chi_2 f, A_\varepsilon(\chi_2 f)\Big\rangle_{L^2(0,\infty)} \\ &\equiv \Big\langle \chi_1 f, A_\varepsilon(\chi_1 f)\Big\rangle_{L^2(0,b)} + \Big\langle \chi_2 f, A_\varepsilon(\chi_2 f)\Big\rangle_{L^2(\frac{b}{4},\infty)}.\end{aligned}$$

Using the identity $\|f\|^2_{L^2(0,\infty)} = \|\chi_1 f\|^2_{L^2(0,b)} + \|\chi_2 f\|^2_{L^2(\frac{b}{4},\infty)}$ and the obvious inclusions $\chi_1 f \in C_c^\infty(0,b)$, $\chi_2 f \in C_c^\infty(\frac{b}{4},\infty)$, we apply the min-max principle as follows:

$$\begin{aligned}E_j(A_\varepsilon) + K &= \inf_{\substack{S\subset C_c^\infty(0,\infty)\\ \dim S = j}} \sup_{\substack{f\in S\\ f\neq 0}} \frac{\langle f, A_\varepsilon f\rangle + K\|f\|^2_{L^2(0,\infty)}}{\|f\|^2_{L^2(0,\infty)}} \\ &\ge \inf_{\substack{S\subset C_c^\infty(0,\infty)\\ \dim S = j}} \sup_{\substack{f\in S\\ f\neq 0}} \frac{\Big\langle \chi_1 f, A_\varepsilon(\chi_1 f)\Big\rangle_{L^2(0,b)} + \Big\langle \chi_2 f, A_\varepsilon(\chi_2 f)\Big\rangle_{L^2(\frac{b}{4},\infty)}}{\|f\|^2_{L^2(0,\infty)}} \\ &= \inf_{\substack{S\subset C_c^\infty(0,\infty)\\ \dim S = j}} \sup_{\substack{f\in S\\ f\neq 0}} \frac{\Big\langle \chi_1 f, A_\varepsilon(\chi_1 f)\Big\rangle_{L^2(0,b)} + \Big\langle \chi_2 f, A_\varepsilon(\chi_2 f)\Big\rangle_{L^2(\frac{b}{4},\infty)}}{\|\chi_1 f\|^2_{L^2(0,b)} + \|\chi_2 f\|^2_{L^2(\frac{b}{4},\infty)}} \qquad (14) \\ &\ge \inf_{\substack{S\subset C_c^\infty(0,b)\oplus C_c^\infty(\frac{b}{4},\infty)\\ \dim S = j}} \sup_{\substack{(f_1,f_2)\in S\\ (f_1,f_2)\neq 0}} \frac{\Big\langle f_1, A_\varepsilon f_1\Big\rangle_{L^2(0,b)} + \Big\langle f_2, A_\varepsilon f_2\Big\rangle_{L^2(\frac{b}{4},\infty)}}{\|f_1\|^2_{L^2(0,b)} + \|f_2\|^2_{L^2(\frac{b}{4},\infty)}} \\ &= \Lambda_j\Big(M_{\varepsilon,b} \oplus \widetilde{M}_{\varepsilon,\frac{b}{4}}\Big) \ge \min\Big\{\Lambda_j(M_{\varepsilon,b}), \inf\operatorname{spec}\widetilde{M}_{\varepsilon,\frac{b}{4}}\Big\}.\end{aligned}$$

For any $j \in \mathbb{N}$ we have $\Lambda_j(M_{\varepsilon,b}) = E_j(M_{\varepsilon,b})$. At the same time, for any function $f \in C_c^\infty(\frac{b}{4},\infty)$ one has

$$\begin{aligned}\langle f, \widetilde{M}_{\varepsilon,\frac{b}{4}} f\rangle_{L^2(\frac{b}{4},\infty)} &= \int_{\frac{b}{4}}^\infty \left[|f'|^2 + \left(\frac{n^2-2n}{4s^2} - \frac{N_\omega}{\varepsilon s}\right) f^2\right] \mathrm{d}s \\ &\ge -\frac{N_\omega}{\varepsilon}\int_{\frac{b}{4}}^\infty \frac{1}{s} f^2 \,\mathrm{d}s \ge -\frac{4N_\omega}{b\varepsilon}\|f\|^2_{L^2(\frac{b}{4},\infty)},\end{aligned}$$

which gives the lower bound $\inf\operatorname{spec}\widetilde{M}_{\varepsilon,\frac{b}{4}} \ge -\frac{4N_\omega}{b\varepsilon}$. Due to (12) we conclude that if $j \in \mathbb{N}$ is fixed, then one can find some $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ there holds $E_j(A_\varepsilon) + K < \inf\operatorname{spec}\widetilde{M}_{\varepsilon,\frac{b}{4}}$. Then (14) implies $E_j(A_\varepsilon) + K \ge E_j(M_{\varepsilon,b})$. □

2.5 Trace theorem on cones

We will need suitable coordinates on $\partial\Omega_\varepsilon$. Consider again the diffeomorphism

$$X : (0,\infty)\times\mathbb{R}^n \to (0,\infty)\times\mathbb{R}^n, \quad X(s,t) = (s,\varepsilon st), \quad (s,t)\equiv(s,t_1,t_2,\dots,t_n),$$

then $\partial\Omega_\varepsilon = X\Big((0,\infty)\times\partial\omega\Big)\cup\{0\}$. Remark that $\{0\}$ has zero n-dimensional Hausdorff measure and can be neglected in the integration over $\partial\Omega_\varepsilon$.

Lemma 8. *For any $\varepsilon>0$, any measurable $v:\partial\Omega_\varepsilon\to\mathbb{R}$ and $u := v\circ X$ there holds*

$$\varepsilon^{n-1}\int_0^\infty\int_{\partial\omega} s^{n-1}\Big|u(s,t)\Big|\,\mathrm{d}\tau(t)\,\mathrm{d}s \\ \le \int_{\partial\Omega_\varepsilon}|v|\,\mathrm{d}\sigma \le \sqrt{1+R^2\varepsilon^2}\,\varepsilon^{n-1}\int_0^\infty\int_{\partial\omega} s^{n-1}\Big|u(s,t)\Big|\,\mathrm{d}\tau(t)\,\mathrm{d}s$$

with $R := \sup_{t\in\omega}|t|$.

Proof. As usual for the integration over hypersurfaces, it is sufficient to prove the statement for functions supported in images of local charts, then it is extended to general functions using a partition of unity.

Let $U\ni z=(z_1,\dots,z_n)\mapsto\varphi(z)$ be a local chart on $\partial\omega$, then

$$\Phi : (0,\infty)\times U\ni(s,z)\mapsto\Big(s,\varepsilon s\varphi(z)\Big)\equiv X\Big(s,\varphi(z)\Big)\in\partial\Omega_\varepsilon$$

is a local chart on $\partial\Omega_\varepsilon$. If $v|_{\partial\Omega_\varepsilon}$ is supported in the image of Φ, then

$$\int_{\partial\Omega_\varepsilon}|v|\,\mathrm{d}\sigma = \int_0^\infty\int_U\Big|v\Big(\Phi(s,z)\Big)\Big|\,g_\Phi(s,z)\,\mathrm{d}z\,\mathrm{d}s, \quad g_\Phi := \sqrt{\det(D\Phi^TD\Phi)}. \tag{15}$$

We compute

$$(D\Phi^TD\Phi)(s,z) = \begin{pmatrix}1+\varepsilon^2|\varphi(z)|^2 & \varepsilon^2sF(z)^T\\ \varepsilon^2sF(z) & \varepsilon^2s^2G_\varphi(z)\end{pmatrix}, \quad G_\varphi := D\varphi^TD\varphi,$$

$$F(z) := \begin{pmatrix}\Big\langle\varphi(z),\partial_1\varphi(z)\Big\rangle_{\mathbb{R}^n}\\ \vdots\\ \Big\langle\varphi(z),\partial_{n-1}\varphi(z)\Big\rangle_{\mathbb{R}^n}\end{pmatrix} \equiv \frac12\nabla_z\Big|\varphi(z)\Big|^2 \equiv \Big|\varphi(z)\Big|\nabla_z\Big|\varphi(z)\Big|.$$

The matrix G_φ is invertible a.e. (as φ is a local chart), therefore, using well-known formulas for the determinants of block matrices (see e.g. [32]) we obtain

$$\begin{aligned} g_\Phi(s,z)^2 &\equiv \det(D\Phi^TD\Phi)\\ &= \Big(1+\varepsilon^2|\varphi(z)|^2 - \Big\langle\varepsilon^2sF(z),\Big(\varepsilon^2s^2G_\varphi(z)\Big)^{-1}\varepsilon^2sF(z)\Big\rangle\Big)\det\Big(\varepsilon^2s^2G_\varphi(z)\Big)\\ &= \varepsilon^{2(n-1)}s^{2(n-1)}\Big[1+\varepsilon^2|\varphi(z)|^2\Big(1-\Big\langle\nabla_z|\varphi(z)|,G_\varphi(z)^{-1}\nabla_z|\varphi(z)|\Big\rangle\Big)\Big]\det G_\varphi(z). \end{aligned}$$

Consider the function $r:t\mapsto|t|$ on $\partial\omega$, then

$$\Big\langle\nabla_z|\varphi(z)|,G_\varphi(z)^{-1}\nabla_z|\varphi(z)|\Big\rangle = |\nabla^{\partial\omega}r|^2\Big(\varphi(z)\Big), \quad |\varphi(z)|^2 = r^2\Big(\varphi(z)\Big),$$

with $\nabla^{\partial\omega} r$ being the tangential gradient of r along $\partial\omega$. Therefore,

$$g_\Phi(s,z) = \varepsilon^{n-1} s^{n-1} \sqrt{1+\varepsilon^2\rho(\varphi(z))}\, g_\varphi(z), \quad g_\varphi(z) := \sqrt{\det G_\varphi(z)},$$
$$\rho := r^2\Big(1 - |\nabla^{\partial\omega} r|^2\Big) \equiv r^2\Big(|\nabla^{\mathbb{R}^n} r|^2 - |\nabla^{\partial\omega} r|^2\Big) \equiv r^2 |\partial_\nu r|^2$$

with ∂_ν being the normal derivative. Due to $|\partial_\nu r| \le 1$ we have $0 \le \rho \le R^2$ with R from (26). By (15) we obtain

$$\varepsilon^{n-1} \int_0^\infty \int_U s^{n-1} \Big|v\Big(\Phi(s,z)\Big)\Big|\, g_\varphi(z)\, \mathrm{d}z\, \mathrm{d}s$$
$$\le \int_{\partial\Omega_\varepsilon} |v|\, \mathrm{d}\sigma \le \sqrt{1+R^2\varepsilon^2}\,\varepsilon^{n-1} \int_0^\infty \int_U s^{n-1} \Big|v\Big(\Phi(s,z)\Big)\Big|\, g_\varphi(z)\, \mathrm{d}z\, \mathrm{d}s. \quad (16)$$

Using the definition of Φ we obtain $v\Big(\Phi(s,z)\Big) = u\Big(s, \varphi(z)\Big)$ and

$$\int_0^\infty \int_U s^{n-1} \Big|v\Big(\Phi(s,z)\Big)\Big|\, g_\varphi(z)\, \mathrm{d}z\, \mathrm{d}s = \int_0^\infty \int_U s^{n-1} \left|u(s,\varphi(z))\right| g_\varphi(z)\, \mathrm{d}z\, \mathrm{d}s$$
$$\equiv \int_0^\infty \int_{\partial\omega} s^{n-1} |u(s,t)|\, \mathrm{d}\tau(t)\, \mathrm{d}s,$$

and the substitution into (16) gives the sought estimate.

The above computations are classical for the case of smooth $\partial\omega$. In our case $\partial\omega$ is only a Lipschitz manifold, but the formulas are still valid a.e.: we refer to [31] for a detailed discussion. □

Recall that the subsets $C^\infty_{(0,\infty)}(\overline{\Omega}_\varepsilon)$ were defined in (6). The restriction of each function from $C^\infty_{(0,\infty)}(\overline{\Omega}_\varepsilon)$ to $\partial\Omega_\varepsilon$ is a continuous function with compact support, hence it is square integrable.

Proposition 9. *Let $\varepsilon > 0$ be fixed. The linear map*

$$\gamma_0 : C^\infty_{(0,\infty)}(\overline{\Omega}_\varepsilon) \to L^2(\partial\Omega_\varepsilon), \quad \gamma_0 u := u|_{\partial\Omega_\varepsilon},$$

uniquely extends to a bounded linear map from $H^1(\Omega_\varepsilon)$ to $L^2(\partial\Omega_\varepsilon)$. Moreover, for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$\|\gamma_0 u\|^2_{L^2(\partial\Omega_\varepsilon)} \le \delta \|\nabla u\|^2_{L^2(\Omega_\varepsilon)} + C_\delta \|u\|^2_{L^2(\Omega_\varepsilon)} \text{ for any } u \in H^1(\Omega_\varepsilon).$$

Proof. It is sufficient to consider $\varepsilon = 1$ (as general values of ε can be absorbed by taking $\varepsilon\omega$ instead of ω). As $C^\infty_{(0,\infty)}(\overline{\Omega}_1)$ is dense in $H^1(\Omega_1)$ by Proposition 5, it is sufficient to show that for any $\delta > 0$ there exists $C_\delta > 0$ such that for any $u \in C^\infty_{(0,\infty)}(\overline{\Omega}_1)$ there holds

$$\int_{\partial\Omega_1} u^2\, \mathrm{d}\sigma \le \delta \int_{\Omega_1} |\nabla u|^2\, \mathrm{d}x + C_\delta \int_{\Omega_1} u^2\, \mathrm{d}x. \quad (17)$$

We use the spectral analysis of the operators B_r from Subsection 2.3. By Lemma 6 one can find a constant $c > 0$ such that

$$\int_\omega |\nabla v|^2\, \mathrm{d}t - r \int_{\partial\omega} v^2\, \mathrm{d}\tau \ge -(N_\omega r + cr^2) \int_\omega v^2\, \mathrm{d}t \text{ for all } v \in H^1(\omega),\ r > 0.$$

and the inequality can be rewritten as

$$\int_{\partial\omega} v^2 \,\mathrm{d}\tau \le \frac{1}{r}\int_\omega |\nabla v|^2 \,\mathrm{d}t + (N_\omega + cr)\int_\omega v^2 \,\mathrm{d}t \text{ for all } v \in H^1(\omega),\ r > 0. \tag{18}$$

Recall that by Lemma 8 we have for any $u \in C^\infty_{(0,\infty)}(\overline{\Omega}_1)$:

$$\int_{\partial\Omega_1} u^2 \,\mathrm{d}\sigma \le \sqrt{1+R^2}\int_0^\infty s^{n-1}\int_{\partial\omega} u(s,st)^2 \,\mathrm{d}\tau(t)\,\mathrm{d}s.$$

We are going to control the integral over $\partial\omega$ using (18) with $v : t \mapsto u(s,st)$ and $r = r(s)$, which gives

$$\begin{aligned}
&\int_{\partial\Omega_1} u^2 \,\mathrm{d}\sigma \le \sqrt{1+R^2}(I_1 + I_2),\\
&\qquad I_1 := \int_0^\infty \frac{s^{n-1}}{r(s)}\int_\omega |\nabla_t u(s,st)|^2 \,\mathrm{d}t\,\mathrm{d}s,\\
&\qquad I_2 := \int_0^\infty s^{n-1}\Big(N_\omega + cr(s)\Big)\int_\omega u(s,st)^2 \,\mathrm{d}t\,\mathrm{d}s.
\end{aligned} \tag{19}$$

Now we remark that

$$\begin{aligned}
I_1 &= \int_0^\infty \frac{s^{n+1}}{r(s)}\int_\omega |(\nabla_{x'} u)(s,st)|^2 \,\mathrm{d}t\,\mathrm{d}s\\
&= \int_0^\infty \frac{s}{r(s)}\int_{s\omega} |(\nabla_{x'} u)(s,x')|^2 \,\mathrm{d}x'\,\mathrm{d}s \le \int_0^\infty \frac{s}{r(s)}\int_{s\omega} |\nabla u(s,x')|^2 \,\mathrm{d}x'\,\mathrm{d}s.
\end{aligned}$$

Taking $r(s) := \mu s$ with a constant $\mu > 0$ to be chosen later we obtain

$$I_1 \le \frac{1}{\mu}\int_0^\infty \int_{s\omega} |\nabla u(s,x')|^2 \,\mathrm{d}x'\,\mathrm{d}s = \frac{1}{\mu}\int_{\Omega_1} |\nabla u|^2 \,\mathrm{d}x.$$

For the same choice of $r(s)$ one has

$$\begin{aligned}
I_2 &= \int_0^\infty s^{n-1}(N_\omega + c\mu s)\int_\omega u(s,st)^2 \,\mathrm{d}t\,\mathrm{d}s\\
&= \underbrace{N_\omega \int_0^\infty s^{n-1}\int_\omega u(s,st)^2 \,\mathrm{d}t\,\mathrm{d}s}_{=:J_1} + c\mu \underbrace{\int_0^\infty s^n \int_\omega u(s,st)^2 \,\mathrm{d}t\,\mathrm{d}s}_{=:J_2}.
\end{aligned}$$

The second term is easy to evaluate:

$$J_2 = \int_0^\infty \int_{s\omega} u(s,x')^2 \,\mathrm{d}x'\,\mathrm{d}s = \int_{\Omega_1} u^2 \,\mathrm{d}x.$$

The term J_1 requires a bit more work. We rewrite

$$J_1 = \int_\omega \int_0^\infty \frac{N_\omega}{s} f_t(s)^2 \,\mathrm{d}s\,\mathrm{d}t \quad \text{with the function} \quad f_t : s \mapsto s^{\frac{n}{2}} u(s,st). \tag{20}$$

For each fixed t one has $f_t \in C_c^\infty(0,\infty)$. Using the spectral analysis of Subsection 2.4 (consider the first eigenvalue of $A_{1/\mu}$ with $n = 2$) we have

$$\int_0^\infty \left[f_t'(s)^2 - \frac{\mu N_\omega}{s} f_t(s)^2\right] \mathrm{d}s \ge -\frac{\mu^2 N_\omega^2}{4}\int_0^\infty f_t(s)^2 \,\mathrm{d}s,$$

which we rewrite as

$$\int_0^\infty \frac{N_\omega}{s} f_t(s)^2 \,\mathrm{d}s \le \frac{1}{\mu}\int_0^\infty f_t'(s)^2 \,\mathrm{d}s + \frac{\mu N_\omega^2}{4}\int_0^\infty f_t(s)^2 \,\mathrm{d}s. \tag{21}$$

We have

$$\begin{aligned} f_t'(s)^2 &= \Big(\tfrac{n}{2} s^{\frac{n}{2}-1} u(s,st) + s^{\frac{n}{2}} \partial_s u(s,st)\Big)^2 \\ &= \tfrac{n^2}{4} s^{n-2} u(s,st)^2 + n s^{n-1} u(s,st)\partial_s u(s,st) + s^n \Big|\partial_s u(s,st)\Big|^2. \end{aligned}$$

Using

$$\begin{aligned} \int_0^\infty n s^{n-1} u(s,st)\partial_s u(s,st)\,\mathrm{d}s &= \frac{n}{2}\int_0^\infty s^{n-1}\partial_s\Big(u(s,st)^2\Big)\,\mathrm{d}s \\ &= -\frac{n}{2}\,(n-1)\int_0^\infty s^{n-2} u(s,st)^2 \,\mathrm{d}s \end{aligned}$$

we arrive at

$$\int_0^\infty f_t'(s)^2 \,\mathrm{d}s = \Big[\frac{n^2}{4} - \frac{n}{2}\,(n-1)\Big]\int_0^\infty s^{n-2} u(s,st)^2 \,\mathrm{d}s + \int_0^\infty s^n \Big|\partial_s u(s,st)\Big|^2 \,\mathrm{d}s.$$

Using $n \ge 2$ one obtains

$$\frac{n^2}{4} - \frac{n}{2}\,(n-1) = \frac{n}{4}\Big(n - 2(n-1)\Big) = \frac{n}{4}\,(2-n) \le 0,$$

which gives

$$\int_0^\infty f_t'(s)^2 \,\mathrm{d}s \le \int_0^\infty s^n \Big|\partial_s u(s,st)\Big|^2 \,\mathrm{d}s.$$

We compute (with the same $R := \sup_{t\in\omega} |t|$ as above)

$$\begin{aligned} \Big|\partial_s u(s,st)\Big|^2 &= \Big|\partial_{x_1} u(s,st) + t\cdot\nabla_{x'} u(s,st)\Big|^2 \le 2|\partial_{x_1} u(s,st)|^2 + 2\Big|t\cdot\nabla_{x'} u(s,st)\Big|^2 \\ &\le 2|\partial_{x_1} u(s,st)|^2 + 2R^2\Big|\nabla_{x'} u(s,st)\Big|^2 \le 2(1+R^2)|(\nabla u)(s,st)|^2, \end{aligned}$$

which results in

$$\int_0^\infty f_t'(s)^2 \,\mathrm{d}s \le 2(1+R^2)\int_0^\infty s^n |(\nabla u)(s,st)|^2 \,\mathrm{d}s.$$

The substitution into (21) gives

$$\int_0^\infty \frac{N_\omega}{s} f_t(s)^2 \,\mathrm{d}s \le \frac{2(1+R^2)}{\mu}\int_0^\infty s^n |(\nabla u)(s,st)|^2 \,\mathrm{d}s + \frac{\mu N_\omega^2}{4}\int_0^\infty s^n u(s,st)^2 \,\mathrm{d}s,$$

and using (20) one obtains

$$\begin{aligned} J_1 &\le \frac{2(1+R^2)}{\mu}\int_0^\infty\int_\omega s^n |(\nabla u)(s,st)|^2 \,\mathrm{d}s\,\mathrm{d}t + \frac{\mu N_\omega^2}{4}\int_0^\infty\int_\omega s^n u(s,st)^2 \,\mathrm{d}t\,\mathrm{d}s \\ &= \frac{2(1+R^2)}{\mu}\int_0^\infty\int_{s\omega} |\nabla u(s,x')|^2 \,\mathrm{d}x'\,\mathrm{d}t + \frac{\mu N_\omega^2}{4}\int_0^\infty\int_{s\omega} u(s,x')^2 \,\mathrm{d}x'\,\mathrm{d}s \\ &= \frac{2(1+R^2)}{\mu}\int_{\Omega_1} |\nabla u|^2 \,\mathrm{d}x + \frac{\mu N_\omega^2}{4}\int_{\Omega_1} u^2 \,\mathrm{d}x. \end{aligned}$$

Using the above estimates for J_1 and J_2 one obtains:

$$I_2 \le \frac{2(1+R^2)}{\mu}\int_{\Omega_1}|\nabla u|^2\,\mathrm{d}x + \mu\Big(\frac{N_\omega^2}{4}+c\Big)\int_{\Omega_1}u^2\,\mathrm{d}x,$$

and the substitution into (19) gives

$$\int_{\partial\Omega_1}u^2\,\mathrm{d}\sigma \le \sqrt{1+R^2}\frac{2(1+R^2)+1}{\mu}\int_{\Omega_1}|\nabla u|^2\,\mathrm{d}x + \sqrt{1+R^2}\,\mu\Big(\frac{N_\omega^2}{4}+c\Big)\int_{\Omega_1}u^2\,\mathrm{d}x.$$

For any $\delta > 0$ one can take μ sufficiently large, such that the coefficient in front of the first integral becomes smaller than δ, and this proves the required inequality (17). □

As an easy corollary we obtain that our spectral problem is well-posed:

Corollary 10. *The bilinear form q_ε is semibounded from below and closed for any $\varepsilon > 0$.*

2.6 Scheme of the proof

We will mostly deal with a Robin Laplacian on a finite part of Ω_ε. Pick some $a > 0$ (this value will remain fixed through the whole text), and denote

$$\begin{aligned} V_\varepsilon &:= \Omega_\varepsilon \cap \{x_1 < a\} \equiv \Big\{(x_1,x') \in (0,a)\times\mathbb{R}^n : x' \in \varepsilon x_1\omega\Big\} \subset \mathbb{R}^{n+1},\\ \partial_0 V_\varepsilon &:= \partial\Omega_\varepsilon \cap \{x_1 < a\} \equiv \Big\{(x_1,x') \in (0,a)\times\mathbb{R}^n : x' \in \varepsilon x_1\partial\omega\Big\} \subset \partial V_\varepsilon,\\ \widehat{H}^1_0(V_\varepsilon) &:= \text{the closure of } C^\infty_{(0,a)}(\overline{\Omega}_\varepsilon) \text{ in } H^1(V_\varepsilon). \end{aligned}$$

Recall that $C^\infty_{(0,a)}(\overline{\Omega}_\varepsilon)$ was defined in (6).

Let T_ε be the self-adjoint operator in $L^2(V_\varepsilon)$ associated with the symmetric bilinear form

$$t_\varepsilon(u,u) = \int_{V_\varepsilon}|\nabla u|^2\,\mathrm{d}x - \int_{\partial_0 V_\varepsilon}u^2\,\mathrm{d}\sigma, \quad D(t_\varepsilon) = \widehat{H}^1_0(V_\varepsilon), \tag{22}$$

then T_ε can be informally interpreted as the Laplacian in V_ε with the Robin boundary condition $\partial_\nu u = u$ on $\partial_0 V_\varepsilon$ and the Dirichlet boundary condition on the remaining boundary $\partial V_\varepsilon \setminus \partial_0 V_\varepsilon$ (which corresponds to $x_1 = a$).

The main part of our analysis is dedicated to the eigenvalues of T_ε (Section 3). Using a suitable change of coordinates and the spectral analysis of B_r, the study of eigenvalues of T_ε with small ε is reduced to the truncated one-dimensional operators $M_{\varepsilon',a}$ (with suitable $\varepsilon' \sim \varepsilon$) from Subsection 2.4. The main result of this reduction is given in Proposition 16. The analysis is in the spirit of the Born-Oppenheimer approximation, see e.g. [29, Part 3], with $M_{\varepsilon',a}$ being an "effective operator", and it is essentially an adaptation of the constructions of the earlier paper [22] on Robin eigenvalues in domains with peaks. We then show in Proposition 20 that the eigenvalues of Q_ε are close to those of T_ε, which finishes the proof of Theorem 1. In view of Proposition 5 the variational eigenvalues of T_ε are defined by

$$\Lambda_j(T_\varepsilon) = \inf_{\substack{S\subset D_0(t_\varepsilon)\\ \dim S = j}}\ \sup_{\substack{u\in S\\ u\neq 0}}\frac{t_\varepsilon(u,u)}{\|u\|^2_{L^2(V_\varepsilon)}}, \quad D_0(t_\varepsilon) := C^\infty_{(0,a)}(\overline{\Omega}_\varepsilon), \quad j\in\mathbb{N}. \tag{23}$$

3 Spectral analysis near the vertex

In this section we study $\Lambda_j(T_\varepsilon)$ with small ε. The proof will be based on (23) and on a kind of asymptotic separation of variables.

3.1 Change of variables

One observes that

$$V_\varepsilon = X(\Pi), \quad \Pi = (0,a)\times\omega, \quad X(s,t) = (s, \varepsilon s t), \quad (s,t) \equiv (s, t_1, t_2, \dots, t_n) \in \Pi.$$

This induces the unitary transform (change of variables)

$$\mathcal{U} : L^2(V_\varepsilon) \to L^2(\Pi, \varepsilon^n s^n \,\mathrm{d}s\, \mathrm{d}t), \quad \mathcal{U}\, u := u \circ X. \tag{24}$$

Consider the symmetric bi linear form p_ε in $L^2(\Pi, \varepsilon^n s^n \,\mathrm{d}s\, \mathrm{d}t)$ given by

$$p_\varepsilon(u,u) := t_\varepsilon(\mathcal{U}^{-1}u, \mathcal{U}^{-1}u), \quad D(p_\varepsilon) = \mathcal{U} D(t_\varepsilon).$$

Due to the unitarity of $\mathcal{U}$ and Proposition 5, the subspace

$$\begin{aligned} D_0(p_\varepsilon) &:= \mathcal{U}\, D_0(t_\varepsilon) \\ &\equiv \Big\{ u \in C^\infty(\overline{\Pi}) : \exists\, [b,c] \subset (0,a) \text{ such that } u(s,t) = 0 \text{ for } s \notin [b,c] \Big\}, \end{aligned}$$

is a core of p_ε, and by (23) one has

$$\Lambda_j(T_\varepsilon) = \inf_{\substack{S \subset D_0(p_\varepsilon) \\ \dim S = j}} \sup_{\substack{u \in S \\ u \neq 0}} \frac{p_\varepsilon(u,u)}{\|u\|^2_{L^2(\Pi, \varepsilon^n s^n \,\mathrm{d}s\, \mathrm{d}t)}}. \tag{25}$$

Now we would like to obtain more convenient expressions for $p_\varepsilon(u,u)$.

Lemma 11. *Denote*

$$R := \sup_{t \in \omega} |t|. \tag{26}$$

For any $v \in D_0(t_\varepsilon)$ and $u := \mathcal{U} v \in D_0(p_\varepsilon)$ there holds

$$\begin{aligned} &\varepsilon^n \int_0^a \int_\omega \Big[\Big(1 - nR\varepsilon\Big) |\partial_s u|^2 + \frac{1 - (nR^2\varepsilon^2 + R\varepsilon)}{\varepsilon^2 s^2} |\nabla_t u|^2 \Big] s^n \,\mathrm{d}t\, \mathrm{d}s \\ &\le \int_{V_\varepsilon} |\nabla v|^2 \,\mathrm{d}x \le \varepsilon^n \int_0^a \int_\omega \Big[\Big(1 + nR\varepsilon\Big) |\partial_s u|^2 + \frac{1 + (nR^2\varepsilon^2 + R\varepsilon)}{\varepsilon^2 s^2} |\nabla_t u|^2 \Big] s^n \,\mathrm{d}t\, \mathrm{d}s. \end{aligned}$$

Proof. A standard computation shows that for any $u \in D_0(p_\varepsilon)$ there holds

$$\int_{V_\varepsilon} |\nabla v|^2 \,\mathrm{d}x = \varepsilon^n \int_0^a \int_\omega \langle \nabla u, G\, \nabla u \rangle_{\mathbb{R}^{n+1}} s^n \,\mathrm{d}t\, \mathrm{d}s \tag{27}$$

where G is the $(n+1)\times(n+1)$ matrix given by

$$G = (DX^T\, DX)^{-1} \equiv \begin{pmatrix} 1 + \varepsilon^2 |t|^2 & \varepsilon^2 s\, t \\ \varepsilon^2 s\, t^T & \varepsilon^2 s^2\, \mathbb{1} \end{pmatrix}^{-1}$$

with DX being the Jacobi matrix of X and $\mathbb{1}$ being the $n \times n$ identity matrix. One checks directly that G is a block matrix,

$$G = \begin{pmatrix} 1 & -\dfrac{t}{s} \\ -\dfrac{t^T}{s} & C \end{pmatrix} \quad \text{with} \quad C_{jk} = \begin{cases} \dfrac{1}{\varepsilon^2 s^2} + \dfrac{t_j^2}{s^2} & \text{if} \quad j = k, \\ \dfrac{t_j t_k}{s^2} & \text{if} \quad j \neq k. \end{cases}$$

We would like to estimate the term $\langle \nabla u, G\, \nabla u \rangle_{\mathbb{R}^{n+1}}$ from above and from below using simpler expressions. One obtains

$$\begin{aligned} \langle \nabla u, G\, \nabla u \rangle_{\mathbb{R}^{n+1}} = |\partial_s u|^2 + \frac{1}{\varepsilon^2 s^2} |\nabla_t u|^2 \\ - \frac{2}{s} \sum_{k=1}^n t_k\, \partial_s u\, \partial_{t_k} u + \frac{1}{s^2} \sum_{j,k=1}^n t_j t_k\, \partial_{t_j} u\, \partial_{t_k} u. \end{aligned} \tag{28}$$

Using the standard inequality $2|xy| \leq x^2 + y^2$ and $|t_j| \leq |t| < R$ we estimate

$$\begin{aligned} \left| \frac{2}{s} \sum_{k=1}^n t_k\, \partial_s u\, \partial_{t_k} u \right| &\leq R\varepsilon \sum_{k=1}^n \left| 2\partial_s u \cdot \frac{\partial_{t_k} u}{\varepsilon s} \right| \\ &\leq R\varepsilon \sum_{k=1}^n \left(|\partial_s u|^2 + \frac{|\partial_{t_k} u|^2}{\varepsilon^2 s^2} \right) = nR\varepsilon |\partial_s u|^2 + \frac{R}{\varepsilon s^2} |\nabla_t u|^2, \\ \left| \frac{1}{s^2} \sum_{j,k=1}^n t_j t_k\, \partial_{t_j} u\, \partial_{t_k} u \right| &\leq \frac{R^2}{s^2} \sum_{j,k=1}^n |\partial_{t_j} u\, \partial_{t_k} u| \\ &\leq \frac{R^2}{2s^2} \sum_{j,k=1}^n \left(|\partial_{t_j} u|^2 + |\partial_{t_k} u|^2 \right) = \frac{nR^2}{s^2} |\nabla_t u|^2. \end{aligned}$$

The substitution into (28) gives a two-sided estimate for $\langle \nabla u, G\, \nabla u \rangle_{\mathbb{R}^{n+1}}$, and the substitution into (27) gives the claim. □

By applying Lemmas 11 and 8 to both summands of t_ε in (22) and by adjusting various constants we obtain the following two-sided estimate written in a form adapted for the subsequent analysis:

Proposition 12. *There exist $c > 0$ and $\varepsilon_0 > 0$, with $c\varepsilon_0 < 1$, both independent of the choice of a, such that for any $\varepsilon \in (0, \varepsilon_0)$ and any $u \in D_0(p_\varepsilon)$ there holds*

$$\begin{aligned} p_\varepsilon^-(u,u) \leq p_\varepsilon(u,u) \leq p_\varepsilon^+(u,u), \\ p_\varepsilon^\pm(u,u) := (1 \pm c\varepsilon)\varepsilon^n \int_0^a \int_\omega s^n \left[|\partial_s u|^2 + \frac{1}{\varepsilon^2 s^2} |\nabla_t u|^2 \right] \mathrm{d}t\, \mathrm{d}s \\ - \frac{1}{1 \pm c\varepsilon} \varepsilon^{n-1} \int_0^a \int_{\partial\omega} s^{n-1} u^2\, \mathrm{d}\tau(t)\, \mathrm{d}s. \end{aligned}$$

In particular, by (25) it follows that for each $j \in \mathbb{N}$ and any $\varepsilon \in (0, \varepsilon_0)$ there holds

$$\inf_{\substack{S \subset D_0(p_\varepsilon) \\ \dim S = j}} \sup_{\substack{u \in S \\ u \neq 0}} \frac{p_\varepsilon^-(u,u)}{\|u\|^2_{L^2(\Pi, \varepsilon^n s^n\, \mathrm{d}s\, \mathrm{d}t)}} \leq \Lambda_j(T_\varepsilon) \leq \inf_{\substack{S \subset D_0(p_\varepsilon) \\ \dim S = j}} \sup_{\substack{u \in S \\ u \neq 0}} \frac{p_\varepsilon^+(u,u)}{\|u\|^2_{L^2(\Pi, \varepsilon^n s^n\, \mathrm{d}s\, \mathrm{d}t)}}.$$

3.2 Upper bound for the eigenvalues of T_ε

We are going to compare the eigenvalues of T_ε with those of the truncated one-dimensional operators $M_{\varepsilon',a}$.

Lemma 13. *There exist $c, c', \varepsilon_0 > 0$, with $c\varepsilon_0 < 1$, such that for any $j \in \mathbb{N}$ and any $\varepsilon \in (0, \varepsilon_0)$ there holds $\Lambda_j(T_\varepsilon) \le (1 + c\varepsilon)E_j(M_{(1+c\varepsilon)^2\varepsilon,a}) + c'$.*

Proof. Take first c and ε_0 as in Proposition 12. Define a unitary transform

$$\mathcal{V} : L^2(\Pi) \to L^2(\Pi, \varepsilon^n s^n \,\mathrm{d}s\,\mathrm{d}t), \quad (\mathcal{V}\,u)(s,t) = \varepsilon^{-\frac{n}{2}} s^{-\frac{n}{2}}\, u(s,t),$$

and consider the symmetric bilinear form $r_\varepsilon^+(u,u) := p_\varepsilon^+(\mathcal{V}\,u, \mathcal{V}\,u)$. One easily sees that for any $u \in D_0(r_\varepsilon^+) := \mathcal{V}^{-1}D_0(p_\varepsilon) \equiv D_0(p_\varepsilon)$ there holds

$$\begin{aligned} r_\varepsilon^+(u,u) = (1+c\varepsilon)\int_0^a\int_\omega \left(\left(\partial_s u - \frac{nu}{2s}\right)^2 + \frac{1}{\varepsilon^2 s^2}\,|\nabla_t u|^2\right)\,\mathrm{d}t\,\mathrm{d}s \\ - \frac{1}{(1+c\varepsilon)\varepsilon}\int_0^a \frac{1}{s}\int_{\partial\omega} u^2\,\mathrm{d}\tau(t)\,\mathrm{d}s. \end{aligned}$$

The substitution $u \mapsto \mathcal{V}u$ into the upper bound of Proposition 12 shows that

$$\Lambda_j(T_\varepsilon) \le \inf_{\substack{S\subset D_0(r_\varepsilon^+)\\ \dim S = j}} \sup_{\substack{u\in S\\ u\neq 0}} \frac{r_\varepsilon^+(u,u)}{\|u\|^2_{L^2(\Pi)}}.$$

Using the density, on the right-hand side one can replace r_ε^+ and $D_0(r_\varepsilon^+)$ by the closure $\overline{r_\varepsilon^+}$ and any dense subset $D \subset D(\overline{r_\varepsilon^+})$. By Lemma 4 we can take

$$D = \Big\{u \in H^1(\Pi) :\ \text{there exists } [b,c] \subset (0,a) \text{ such that } u(x) = 0 \text{ for } x_1 \notin [b,c]\Big\},$$

and we keep the symbol r_ε^+ for $\overline{r_\varepsilon^+}$ on D, as it is given by the same expression. Therefore,

$$\Lambda_j(T_\varepsilon) \le \inf_{\substack{S\subset D\\ \dim S = j}} \sup_{\substack{u\in S\\ u\neq 0}} \frac{r_\varepsilon^+(u,u)}{\|u\|^2_{L^2(\Pi)}}, \tag{29}$$

Then integration by parts shows that for $u \in D$ one has

$$\int_\omega\int_0^a \frac{u\partial_s u}{s}\,\mathrm{d}s = \int_\omega\int_0^a \frac{u^2}{2s^2}\,\mathrm{d}s, \tag{30}$$

which implies

$$\begin{aligned} r_\varepsilon^+(u,u) &= (1+c\varepsilon)\int_0^a\int_\omega \left(\left(|\partial_s u|^2 + \frac{n^2-2n}{4s^2}\,u^2\right) + \frac{1}{\varepsilon^2 s^2}\,|\nabla_t u|^2\right)\mathrm{d}t\,\mathrm{d}s \\ &\quad - \frac{1}{(1+c\varepsilon)\varepsilon}\int_0^a \frac{1}{s}\int_{\partial\omega} u^2\,\mathrm{d}\tau\,\mathrm{d}s \\ &= (1+c\varepsilon)\Bigg[\int_0^a\int_\omega \left(|\partial_s u|^2 + \frac{n^2-2n}{4s^2}\,u^2\right)\mathrm{d}t\,\mathrm{d}s \\ &\quad + \int_0^a \frac{1}{\varepsilon^2 s^2}\Big\{\int_\omega |\nabla_t u|^2\,\mathrm{d}t - \frac{\varepsilon s}{(1+c\varepsilon)^2}\int_{\partial\omega} u^2\,\mathrm{d}\tau\Big\}\,\mathrm{d}s\Bigg]. \end{aligned} \tag{31}$$

Note that the functional in the curly brackets is the bilinear form $b_{\varepsilon\rho(s,\varepsilon)}$ as defined in Subsection 2.3 with $\rho(s,\varepsilon) = s(1+c\varepsilon)^{-2}$. Let $\psi \equiv \psi_{\varepsilon\rho(s,\varepsilon)}$ be the positive normalized eigenfunction of $B_{\varepsilon\rho(s,\varepsilon)}$ for $E_1(B_{\varepsilon\rho(s,\varepsilon)})$. By Lemma 6, for any $\varepsilon > 0$ the map $\mathbb{R} \ni s \mapsto \psi_{\varepsilon\rho(s,\varepsilon)} \in L^2(\omega)$ is C^∞. If $f \in C_c^\infty(0,a)$, then also $\mathbb{R} \ni s \mapsto f(s)\psi_{\varepsilon\rho(s,\varepsilon)} \in L^2(\omega)$ is C^∞, and the derivative (which is smooth and with compact support) coincides with the weak derivative in Π with respect to s. It follows that the function $(s,t) \mapsto f(s)\psi_{\varepsilon\rho(s,\varepsilon)}(t)$ belongs to the above subspace D. Moreover, if $S \subset C_c^\infty(0,a)$ is a j-dimensional subspace, then

$$\widetilde{S} = \Big\{u : \Pi \to \mathbb{R} :\ u(s,t) = f(s)\,\psi_{\varepsilon\rho(s,\varepsilon)}(t),\ f \in S\Big\}.$$

is a j-dimensional subspace of D. For any $u \in \widetilde{S}$ one has $\|u\|_{L^2(\Pi)} = \|f\|_{L^2(0,a)}$ by Fubini's theorem and

$$\int_\omega |\nabla_t u|^2 \,\mathrm{d}t - \frac{\varepsilon s}{(1+c\varepsilon)^2}\int_{\partial\omega} u^2 \,\mathrm{d}\tau = E_1(B_{\varepsilon\rho(s,\varepsilon)})f(s)^2$$

due to the spectral theorem. Furthermore,

$$\begin{aligned}\int_0^a\int_\omega |\partial_s u|^2 \,\mathrm{d}t\,\mathrm{d}s &= \int_0^a\int_\omega \Big|f'(s)\psi_{\varepsilon\rho(s,\varepsilon)}(t) + f(s)\partial_s\psi_{\varepsilon\rho(s,\varepsilon)}(t)\Big|^2 \,\mathrm{d}t\,\mathrm{d}s\\ &= \int_0^a\int_\omega \Big[f'(s)^2\psi_{\varepsilon\rho(s,\varepsilon)}(t)^2 + f(s)^2|\partial_s\psi_{\varepsilon\rho(s,\varepsilon)}(t)|^2\\ &\qquad + f(s)f'(s)\cdot 2\psi_{\varepsilon\rho(s,\varepsilon)}(t)\,\partial_s\psi_{\varepsilon\rho(s,\varepsilon)}(t)\Big]\,\mathrm{d}t\,\mathrm{d}s,\end{aligned}$$

while

$$\int_\omega 2\psi_{\varepsilon\rho(s,\varepsilon)}(t)\partial_s\psi_{\varepsilon\rho(s,\varepsilon)}(t)\,\mathrm{d}t = \int_\omega \partial_s\Big|\psi_{\varepsilon\rho(s,\varepsilon)}(t)\Big|^2\,\mathrm{d}t = \partial_s\|\psi_{\varepsilon\rho(s,\varepsilon)}\|^2_{L^2(\omega)} = \partial_s 1 = 0.$$

Therefore,

$$\int_0^a\int_\omega \Big(|\partial_s u|^2 + \frac{n^2-2n}{4s^2}\,u^2\Big)\,\mathrm{d}t\,\mathrm{d}s = \int_0^a \Big[|f'|^2 + \Big(\frac{n^2-2n}{4s^2} + \int_\omega |\partial_s\psi_{\varepsilon\rho(s,\varepsilon)}|^2\,\mathrm{d}t\Big)f^2\Big]\,\mathrm{d}s.$$

The substitution into (31) shows that for any $u \in \widetilde{S}$ there holds

$$r_\varepsilon^+(u,u) = (1+c\varepsilon)\int_0^a \Big[|f'|^2 + \Big(\frac{n^2-2n}{4s^2} + \int_\omega |\partial_s\psi_{\varepsilon\rho(s,\varepsilon)}|^2\,\mathrm{d}t + \frac{E_1(B_{\varepsilon\rho(s,\varepsilon)})}{\varepsilon^2 s^2}\Big)f^2\Big]\,\mathrm{d}s.$$

By the estimate (9) in Lemma 6 we can control the term with $\partial_s\psi$. Namely, for $s \in (0,a)$ and $\varepsilon \in (0,\varepsilon_0)$ the values of $\varepsilon\rho(s,\varepsilon)$ are contained in some bounded interval, and then one can find some $K > 0$ such that

$$\begin{aligned}\int_\omega \Big|\partial_s\psi_{\varepsilon\rho(s,\varepsilon)}(t)\Big|^2\,\mathrm{d}t &= \varepsilon^2\int_\omega \Big(\partial_\rho\psi_\rho(t)\Big|_{\rho=\varepsilon\rho(s,\varepsilon)}\frac{\partial\rho(s,\varepsilon)}{\partial s}\Big)^2\,\mathrm{d}t\\ &= \frac{\varepsilon^2}{(1+c\varepsilon)^4}\int_\omega \Big(\partial_\rho\psi_\rho(t)\Big|_{\rho=\varepsilon\rho(s,\varepsilon)}\Big)^2\,\mathrm{d}t \le K\varepsilon^2.\end{aligned}$$

Hence, for all $u \in \widetilde{S}$ and $\varepsilon \in (0,\varepsilon_0)$ one has

$$r_\varepsilon^+(u,u) \le (1+c\varepsilon)\int_0^a \Big[|f'|^2 + \Big(\frac{n^2-2n}{4s^2} + K\varepsilon^2 + \frac{E_1(B_{\varepsilon\rho(s,\varepsilon)})}{\varepsilon^2 s^2}\Big)f^2\Big]\,\mathrm{d}s.$$

Now we apply Lemma 6(a,c) to the eigenvalue $E_1(B_{\varepsilon\rho(s,\varepsilon)})$: there exists $c_0 > 0$ such that for all $s > 0$ and $\varepsilon > 0$ there holds

$$\frac{E_1(B_{\varepsilon\rho(s,\varepsilon)})}{\varepsilon^2 s^2} \le \frac{-N_\omega\,\varepsilon\,\rho(s,\varepsilon) + c_0\,\varepsilon^2\rho^2(s,\varepsilon)}{\varepsilon^2 s^2} = -\frac{N_\omega}{\varepsilon\,s}\cdot\frac{1}{(1+c\varepsilon)^2} + \frac{c_0}{(1+c\varepsilon)^4}.$$

Hence, for all $u \in \widetilde{S}$ and $\varepsilon \in (0,\varepsilon_0)$ we obtain

$$\begin{aligned}
\frac{r_\varepsilon^+(u,u)}{\|u\|^2_{L^2(\Pi)}} &\le \frac{(1+c\varepsilon)\displaystyle\int_0^a \left[|f'|^2 + \left(\frac{n^2-2n}{4s^2} - \frac{N_\omega}{\varepsilon\,s}\cdot\frac{1}{(1+c\varepsilon)^2}\right) f^2\right] \mathrm{d}s}{\|f\|^2_{L^2(0,a)}} \\
&\quad + \frac{c_0}{(1+c\varepsilon)^3} + (1+c\varepsilon)K\varepsilon^2 \\
&= (1+c\varepsilon)\frac{\langle f, M_{(1+c\varepsilon)^2\varepsilon,a} f\rangle_{L^2(0,a)}}{\|f\|^2_{L^2(0,a)}} + \frac{c_0}{(1+c\varepsilon)^3} + (1+c\varepsilon)K\varepsilon^2 \\
&\le (1+c\varepsilon)\frac{\langle f, M_{(1+c\varepsilon)^2\varepsilon,a} f\rangle_{L^2(0,a)}}{\|f\|^2_{L^2(0,a)}} + c' \text{ for } c' := c_0 + (1+c\varepsilon_0)^2 K\varepsilon_0^2.
\end{aligned}$$

The constants c, c', ε_0 are independent of j and S. By (29), for any $j \in \mathbb{N}$ and $\varepsilon \in (0,\varepsilon_0)$ there holds

$$\begin{aligned}
\Lambda_j(T_\varepsilon) &\le \inf_{\substack{S\subset D\\ \dim S = j}} \sup_{\substack{u\in S\\ u\neq 0}} \frac{r_\varepsilon^+(u,u)}{\|u\|^2_{L^2(\Pi)}} \le \inf_{\substack{S\subset C_c^\infty(0,a)\\ \dim S = j}} \sup_{\substack{u\in \widetilde{S}\\ u\neq 0}} \frac{r_\varepsilon^+(u,u)}{\|u\|^2_{L^2(\Pi)}} \\
&\le (1+c\varepsilon) \inf_{\substack{S\subset C_c^\infty(0,a)\\ \dim S = j}} \sup_{\substack{f\in S\\ f\neq 0}} \frac{\langle f, M_{(1+c\varepsilon)^2\varepsilon,a} f\rangle_{L^2(0,a)}}{\|f\|^2_{L^2(0,a)}} + c' \\
&= (1+c\varepsilon)E_j(M_{(1+c\varepsilon)^2\varepsilon,a}) + c'.
\end{aligned}$$

□

Corollary 14. *For any $j \in \mathbb{N}$ there exist $k > 0$ and $\varepsilon_0 > 0$ such that*

$$\Lambda_j(T_\varepsilon) \le -\frac{N_\omega^2}{(2j+n-2)^2\varepsilon^2} + \frac{k}{\varepsilon} \text{ for all } \varepsilon \in (0,\varepsilon_0).$$

Proof. By Lemma 7 for any fixed $c > 0$ and $j \in \mathbb{N}$ we can choose $K' > 0$ such that

$$E_j(M_{(1+c\varepsilon)^2\varepsilon,a}) \le E_j(A_{(1+c\varepsilon)^2\varepsilon}) + K' \equiv -\frac{N_\omega^2}{(2j+n-2)^2\varepsilon^2}\cdot\left(\frac{1}{1+c\varepsilon}\right)^4 + K'$$

if ε is small enough. The substitution into Lemma 13 gives the result. □

3.3 Lower bound for the eigenvalues of T_ε

The lower bound for the eigenvalues of T_ε is also obtained using a comparison with the operators $M_{\varepsilon',a}$ but requires more work.

Lemma 15. *Let $j \in \mathbb{N}$, then there exist $\varepsilon_0 > 0$ and $k' > 0$ such that*

$$\Lambda_j(T_\varepsilon) \ge -\frac{N_\omega^2}{(2j+n-2)^2\varepsilon^2} - \frac{k'}{\varepsilon} \text{ for all } \varepsilon \in (0,\varepsilon_0).$$

Proof. Take c and ε_0 as in Proposition 12. For $\varepsilon \in (0, \varepsilon_0)$ consider again the unitary transform $\mathcal{V} : L^2(\Pi) \to L^2(\Pi, \varepsilon^n s^n \,\mathrm{d}s\, \mathrm{d}t)$, $(\mathcal{V} u)(s,t) = \varepsilon^{-\frac{n}{2}} s^{-\frac{n}{2}} u(s,t)$, and the symmetric bilinear form $r_\varepsilon^-(u,u) := p_\varepsilon^-(\mathcal{V} u, \mathcal{V} u)$. The reparametrization $u \mapsto \mathcal{V} u$ in the lower bound of Proposition 12 leads to

$$\Lambda_j(T_\varepsilon) \geq \inf_{\substack{S \subset D_0(r_\varepsilon^-) \\ \dim S = j}} \sup_{\substack{u \in S \\ u \neq 0}} \frac{r_\varepsilon^-(u,u)}{\|u\|^2_{L^2(\Pi)}}, \quad D_0(r_\varepsilon^-) := \mathcal{V}^{-1} D_0(p_\varepsilon) \equiv D_0(p_\varepsilon). \tag{32}$$

The substitution of $\mathcal{V} u$ into p_ε^- and the partial integration (30) show that

$$\begin{aligned} r_\varepsilon^-(u,u) &= (1 - c\varepsilon) \int_0^a \int_\omega \left(\left(|\partial_s u|^2 + \frac{n^2 - 2n}{4s^2}\, u^2 \right) + \frac{1}{\varepsilon^2 s^2} \, |\nabla_t u|^2 \right) \mathrm{d}t \, \mathrm{d}s \\ &\quad - \frac{1}{(1 - c\varepsilon)\varepsilon} \int_0^a \frac{1}{s} \int_{\partial\omega} u^2 \, \mathrm{d}\tau \, \mathrm{d}s \\ &= (1 - c\varepsilon) \Bigg[\int_0^a \int_\omega \left(|\partial_s u|^2 + \frac{n^2 - 2n}{4s^2}\, u^2 \right) \mathrm{d}t \, \mathrm{d}s \\ &\quad + \int_0^a \frac{1}{\varepsilon^2 s^2} \Big\{ \int_\omega |\nabla_t u|^2 \, \mathrm{d}t - \frac{\varepsilon s}{(1 - c\varepsilon)^2} \int_{\partial\omega} u^2 \, \mathrm{d}\tau \Big\} \, \mathrm{d}s \Bigg], \\ \rho(s,\varepsilon) &:= \frac{s}{(1 - c\varepsilon)^2} \in (0, m), \quad m := \frac{a}{(1 - c\varepsilon_0)^2}, \quad \varepsilon \in (0, \varepsilon_0). \end{aligned} \tag{33}$$

The expression in the curly brackets is the bilinear form $b_{\varepsilon\rho(s,\varepsilon)}$ for the Robin Laplacian $B_{\varepsilon\rho(s,\varepsilon)}$ on ω as discussed in Subsection 2.3. Denote by $\psi_{\varepsilon\rho(s,\varepsilon)}$ the positive eigenfunction for $E_1(B_{\varepsilon\rho(s,\varepsilon)})$ with $\|\psi_{\varepsilon\rho(s,\varepsilon)}\|_{L^2(\omega)} = 1$, then $s \mapsto \psi_{\varepsilon\rho(s,\varepsilon)}$ is C^∞ by Lemma 6. We decompose each $u \in D_0(r_\varepsilon^-)$ as

$$u = v + w \quad \text{with} \quad v(s,t) = \psi_{\varepsilon\rho(s,\varepsilon)}(t)\, f(s), \quad f(s) := \int_\omega u(s,t)\, \psi_{\varepsilon\rho(s,\varepsilon)}(t) \, \mathrm{d}t.$$

By construction we have $f \in C_c^\infty(0,a)$ and, furthermore,

$$\int_\omega w(s,t)\, \psi_{\varepsilon\rho(s,\varepsilon)}(t) \, \mathrm{d}t = 0 \quad \text{for any } s \in (0,a), \tag{34}$$

$$\|f\|_{L^2(0,a)} = \|v\|_{L^2(\Pi)}, \quad \|f\|^2_{L^2(0,a)} + \|w\|^2_{L^2(\Pi)} = \|u\|^2_{L^2(\Pi)}. \tag{35}$$

The spectral theorem applied to $B_{\varepsilon\rho(s,\varepsilon)}$ implies that for any $u \in D_0(r_\varepsilon^-)$ there holds

$$\begin{aligned} \int_\omega |\nabla_t u(s,t)|^2 \, \mathrm{d}t - \varepsilon\rho(s,\varepsilon) \int_{\partial\omega} u(s,t)^2 \, \mathrm{d}\tau(t) \\ \geq E_1(B_{\varepsilon\rho(s,\varepsilon)})\, f(s)^2 + E_2(B_{\varepsilon\rho(s,\varepsilon)}) \int_\omega w(s,t)^2 \, \mathrm{d}t\,. \end{aligned} \tag{36}$$

By Lemma 6(c) one can find a constant $c_1 > 0$ such that

$$E_1(B_x) = -N_\omega\, x + O(x^2) > -\frac{N_\omega\, x}{1 - c_1 x} \quad \text{for all sufficiently small } x > 0.$$

We have $\varepsilon\rho(s,\varepsilon) \in [0, m\varepsilon_0]$. By adjusting the value of ε_0 we conclude that there exists $c_2 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ and $s \in (0,a)$ one has

$$\begin{aligned}\frac{E_1(B_{\varepsilon\rho(s,\varepsilon)})}{\varepsilon^2 s^2} &\geq -\frac{N_\omega\,\varepsilon\rho(s,\varepsilon)}{\varepsilon^2 s^2\Big(1 - c_1\varepsilon\rho(s,\varepsilon)\Big)} = -\frac{N_\omega\,\varepsilon s}{(1-c\varepsilon)^2\varepsilon^2 s^2\Big(1 - \dfrac{c_1\varepsilon s}{(1-c\varepsilon)^2}\Big)} \\ &= -\frac{N_\omega}{\varepsilon s\Big((1-c\varepsilon)^2 - c_1\varepsilon s\Big)} \geq -\frac{N_\omega}{\varepsilon s(1-c_2\varepsilon)}.\end{aligned}$$

By Lemma 6(d) we can find $C_0 > 0$ such that $E_2(B_x) = E_2^N + o(1) \geq C_0$ for small $x > 0$. Hence, if ε_0 is sufficiently small, $s \in (0,a)$ and $\varepsilon \in (0,\varepsilon_0)$, then (36) implies

$$\frac{1}{\varepsilon^2 s^2}\left(\int_\omega |\nabla_t u|^2\,\mathrm{d}t - \rho(s,\varepsilon)\int_{\partial\omega} u^2\,\mathrm{d}\tau\right) \;\geq\; -\frac{N_\omega}{\varepsilon s(1-c_2\varepsilon)} f(s)^2 + \frac{C_0}{\varepsilon^2 s^2}\int_\omega w^2\,\mathrm{d}t\,,$$

which is valid for all $u \in D_0(r_\varepsilon^-)$. The substitution of the last inequality into (33) shows that for all $\varepsilon \in (0,\varepsilon_0)$ and $u \in D_0(r_\varepsilon^-)$ there holds

$$\begin{aligned}r_\varepsilon^-(u,u) \geq{}& (1-c\varepsilon)\int_0^a\int_\omega\left(|\partial_s u|^2 + \frac{n^2-2n}{4s^2}\,u^2\right)\mathrm{d}t\,\mathrm{d}s \\ &+ (1-c\varepsilon)C_0\int_0^a\int_\omega \frac{w^2}{\varepsilon^2 s^2}\,\mathrm{d}t\,\mathrm{d}s - N_\omega\frac{1-c\varepsilon}{1-c_2\varepsilon}\int_0^a \frac{f^2}{\varepsilon s}\,\mathrm{d}s.\end{aligned}$$

To have a simpler writing we further choose a suitable $k > c$ and adjust ε_0 such that for all $\varepsilon \in (0,\varepsilon_0)$ and $u \in D_0(r_\varepsilon^-)$ one obtains

$$\begin{aligned}r_\varepsilon^-(u,u) \geq{}& (1-k\varepsilon)\int_0^a\int_\omega\left(|\partial_s u|^2 + \frac{n^2-2n}{4s^2}\,u^2\right)\mathrm{d}t\,\mathrm{d}s \\ &+ \frac{C_0}{2}\int_0^a\int_\omega \frac{w^2}{\varepsilon^2 s^2}\,\mathrm{d}t\,\mathrm{d}s - \frac{N_\omega}{1-k\varepsilon}\int_0^a \frac{f^2}{\varepsilon s}\,\mathrm{d}s.\end{aligned} \tag{37}$$

For the sake of brevity we will denote

$$\psi := \psi_{\varepsilon\rho(s,\varepsilon)}, \quad \psi_s := \partial_s\psi, \quad v_s := \partial_s v, \quad w_s := \partial_s w\,.$$

Let us study the first integral on the right hand side of (37). Using the orthogonality relations (34) we obtain

$$\begin{aligned}\int_0^a\int_\omega\left(|\partial_s u|^2 + \frac{n^2-2n}{4s^2}\,u^2\right)\mathrm{d}t\,\mathrm{d}s &= \int_0^a\!\!\int_\omega\left(v_s^2 + \frac{n^2-2n}{4s^2}\,v^2\right)\mathrm{d}t\,\mathrm{d}s \\ &+ \int_0^a\!\!\int_\omega\left(w_s^2 + \frac{n^2-2n}{4s^2}\,w^2\right)\mathrm{d}t\,\mathrm{d}s + 2\int_0^a\!\!\int_\omega v_s\,w_s\;\mathrm{d}t\;\mathrm{d}s.\end{aligned} \tag{38}$$

We have

$$\begin{aligned}\int_0^a\int_\omega v_s^2\,\mathrm{d}t\,\mathrm{d}s &= \int_0^a\int_\omega |f'\psi + f\psi_s|^2\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_0^a\int_\omega\Big(|f'|^2\psi^2 + f^2|\psi_s|^2 + 2ff'\psi_s\psi\Big)\,\mathrm{d}t\,\mathrm{d}s.\end{aligned}$$

Due to the normalization of ψ one has

$$\int_\omega 2\psi\psi_s \,\mathrm{d}t = \partial_s \int_\omega \psi^2 \,\mathrm{d}t = \partial_s 1 = 0,$$

therefore,

$$\int_0^a \int_\omega v_s^2 \,\mathrm{d}t\,\mathrm{d}s = \int_0^a \left(|f'|^2 + \|\psi_s\|^2_{L^2(\omega)}|f|^2\right)\mathrm{d}s \geq \int_0^a |f'|^2 \,\mathrm{d}s$$

and, consequently,

$$\int_0^a \int_\omega \left(v_s^2 + \frac{n^2-2n}{4s^2}\, v^2\right)\mathrm{d}t\,\mathrm{d}s \geq \int_0^a \left[|f'|^2 + \frac{n^2-2n}{4s^2}\, f^2\right]\mathrm{d}s. \tag{39}$$

In order to estimate the two last terms in (38) we note that

$$2\int_0^a \int_\omega v_s\, w_s \;\mathrm{d}t\,\mathrm{d}s = 2\int_0^a \int_\omega f'\,\psi\, w_s \;\mathrm{d}t\,\mathrm{d}s + 2\int_0^a f \int_\omega \psi_s\, w_s \;\mathrm{d}t\;\mathrm{d}s \tag{40}$$

and that, in view of (34),

$$\int_\omega (\psi w_s + \psi_s w)\,\mathrm{d}t = \partial_s \int_\omega \psi w\,\mathrm{d}t = \partial_s 0 = 0, \quad \int_\omega \psi\, w_s \;\mathrm{d}t = -\int_\omega \psi_s w \;\mathrm{d}t.$$

Hence, using $|2xy| \leq x^2 + y^2$,

$$\begin{aligned}\left|2\int_0^a \int_\omega f'\,\psi\, w_s \;\mathrm{d}t\,\mathrm{d}s\right| &= \left|2\int_0^a \int_\omega f'\,\psi_s\, w \;\mathrm{d}t\;\mathrm{d}s\right| \\ &\leq \int_0^a \int_\omega \left(|f'|^2\psi_s^2 + w^2\right)\mathrm{d}t\,\mathrm{d}s = \int_0^a \|\psi_s\|^2_{L^2(\omega)}|f'|^2\,\mathrm{d}s + \|w\|^2_{L^2(\Pi)}.\end{aligned} \tag{41}$$

Similarly,

$$\begin{aligned}\left|2\int_0^a f \int_\omega \psi_s\, w_s\,\mathrm{d}t\,\mathrm{d}s\right| &= \left|2\int_0^a \int_\omega f\cdot\frac{1}{\sqrt{\varepsilon}}\psi_s\,\sqrt{\varepsilon} w_s\,\mathrm{d}t\,\mathrm{d}s\right| \\ &\leq \int_0^a \int_\omega \left(f^2\cdot\frac{1}{\varepsilon}\psi_s^2 + \varepsilon w_s^2\right)\mathrm{d}t\,\mathrm{d}s = \frac{1}{\varepsilon}\int_0^a f^2\|\psi_s\|^2_{L^2(\omega)}\,\mathrm{d}s + \varepsilon\|w_s\|^2_{L^2(\Pi)}.\end{aligned} \tag{42}$$

Now we represent

$$\|\psi_s\|^2_{L^2(\omega)} = \int_\omega |\partial_s\psi_{\varepsilon\rho(s,\varepsilon)}(t)|^2\,\mathrm{d}t = \frac{\varepsilon^2}{(1-c\varepsilon)^4}\int_\omega \left(\partial_\rho\psi_\rho(t)\Big|_{\rho=\varepsilon\rho(s,\varepsilon)}\right)^2\mathrm{d}t.$$

As $\varepsilon\rho(s,\varepsilon) \in (0,\varepsilon_0 m)$ for all $s \in (0,a)$ and $\varepsilon \in (0,\varepsilon_0)$ we can use the estimate (9) of Lemma 6: there exists $K > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ and $s \in (0,a)$ one has $\|\psi_s\|^2_{L^2(\omega)} \leq K\varepsilon^2 \leq \varepsilon$ (assuming that ε_0 is sufficiently small). We now use the obtained estimate in (41) and (42), which gives

$$\begin{aligned}\left|\int_0^a \int_\omega f'\,\psi\, w_s \;\mathrm{d}t\,\mathrm{d}s\right| &\leq \varepsilon\int_0^a |f'|^2 \;\mathrm{d}s + \|w\|^2_{L^2(\Pi)}, \\ \left|\int_0^a f\int_\omega \psi_s\, w_s \;\mathrm{d}t\;\mathrm{d}s\right| &\leq K\varepsilon\int_0^a f^2 \;\mathrm{d}s + \varepsilon\|w_s\|^2_{L^2(\Pi)}.\end{aligned}$$

The substitution of these two inequalities into (40) gives

$$\Big|\int_0^a\int_\omega v_s\,w_s\;\mathrm{d}t\,\mathrm{d}s\Big| \le \varepsilon\int_0^a |f'|^2\;\mathrm{d}s + K\varepsilon\int_0^a f^2\;\mathrm{d}s + \|w\|^2_{L^2(\Pi)} + \varepsilon\|w_s\|^2_{L^2(\Pi)}.$$

We now use the last obtained inequality and (39) in (38), which gives

$$\begin{aligned}\int_0^a\int_\omega\Big(|\partial_s u|^2 + \frac{n^2-2n}{4s^2}\,u^2\Big)\;\mathrm{d}t\,\mathrm{d}s \ge \int_0^a\Big[(1-\varepsilon)\,|f'|^2 + \Big(\frac{n^2-2n}{4s^2} - K\varepsilon\Big)f^2\Big]\;\mathrm{d}s\\ + \int_0^a\int_\omega\Big[(1-\varepsilon)\,w_s^2 + \Big(\frac{n^2-2n}{4s^2}-1\Big)w^2\Big]\;\mathrm{d}t\,\mathrm{d}s\,,\end{aligned}$$

for all $\varepsilon\in(0,\varepsilon_0)$ and $u\in D_0(r_\varepsilon^-)$. Using this lower bound in (37) one arrives at

$$\begin{aligned}r_\varepsilon^-(u,u) &\ge (1-k\varepsilon)\int_0^a\Big[(1-\varepsilon)\,|f'|^2 + \Big(\frac{n^2-2n}{4s^2} - K\varepsilon\Big)\,f^2\Big]\;\mathrm{d}s\\ &\quad + (1-k\varepsilon)\int_0^a\int_\omega\Big[(1-\varepsilon)w_s^2 + \Big(\frac{n^2-2n}{4s^2}-1\Big)\,w^2\Big]\;\mathrm{d}t\,\mathrm{d}s\\ &\quad + \frac{C_0}{2}\int_0^a\int_\omega\frac{w^2}{\varepsilon^2 s^2}\;\mathrm{d}t\,\mathrm{d}s - \frac{N_\omega}{1-k\varepsilon}\int_0^a\frac{f^2}{\varepsilon s}\,\mathrm{d}s\\ &\ge (1-k\varepsilon)\int_0^a\Big[(1-\varepsilon)\,|f'|^2 + \Big(\frac{n^2-2n}{4s^2} - K\varepsilon\Big)\,f^2\Big]\;\mathrm{d}s\\ &\quad - \frac{N_\omega}{1-k\varepsilon}\int_0^a\frac{f^2}{\varepsilon s}\,\mathrm{d}s - \|w\|^2_{L^2(\Pi)}.\end{aligned}$$

By taking sufficiently large $b>k$ and $c'>1$ and a smaller value of ε_0 one deduces from the last inequality the simpler lower bound

$$\begin{aligned}r_\varepsilon^-(u,u) &\ge (1-b\varepsilon)\int_0^a\Big[|f'|^2 + \Big(\frac{n^2-2n}{4s^2} - \frac{N_\omega}{(1-b\varepsilon)^2\varepsilon s}\Big)f^2\Big]\,\mathrm{d}s\\ &\quad - c'\|f\|^2_{L^2(0,a)} - c'\|w\|^2_{L^2(\Pi)}.\end{aligned}$$

Using the norm equality (35) this is equivalent to

$$\begin{aligned}r_\varepsilon^-(u,u) + c'\|u\|^2_{L^2(\Pi)} &\ge (1-b\varepsilon)\int_0^a\Big[|f'|^2 + \Big(\frac{n^2-2n}{4s^2} - \frac{N_\omega}{(1-b\varepsilon)^2\varepsilon s}\Big)f^2\Big]\,\mathrm{d}s\\ &\equiv (1-b\varepsilon)\langle f, M_{(1-b\varepsilon)^2\varepsilon,a}f\rangle_{L^2(0,a)}.\end{aligned}\tag{43}$$

By the norm equality (35), the map $u\mapsto(f,w)$ uniquely extends to a unitary map $\Psi: L^2(\Pi)\to L^2(0,a)\oplus\mathcal{H}$, where $\mathcal{H}$ is some closed subspace of $L^2(\Pi)$. Let h_ε be the symmetric bilinear form in $L^2(0,a)\oplus\mathcal{H}$ defined as the closure of the form

$$C_c^\infty(0,a)\times\mathcal{H}\ni(f,w)\mapsto\int_0^a\Big[\,|f'|^2 + \Big(\frac{n^2-2n}{4s^2} - \frac{N_\omega}{(1-b\varepsilon)^2\varepsilon s}\Big)\,f^2\Big]\;\mathrm{d}s,$$

then the corresponding self-adjoint operator in $L^2(0,a)\oplus\mathcal{H}$ is $H_\varepsilon = M_{(1-b\varepsilon)^2\varepsilon,a}\oplus 0$. The inequality (43) reads as $r_\varepsilon^-(u,u) + c'\|u\|^2_{L^2(\Pi)} \ge (1-b\varepsilon)\,h_\varepsilon(\Psi u,\Psi u)$ for all

$u \in D_0(r_\varepsilon^-)$, and the lower bound (32) for $\Lambda_j(T_\varepsilon)$ implies that for any $j \in \mathbb{N}$ and any $\varepsilon \in (0, \varepsilon_0)$ there holds

$$\begin{aligned}\Lambda_j(T_\varepsilon) + c' &\geq \inf_{\substack{S \subset D_0(r_\varepsilon^-) \\ \dim S = j}} \sup_{\substack{u \in S \\ u \neq 0}} \frac{r_\varepsilon^-(u,u) + c'\|u\|^2_{L^2(\Pi)}}{\|u\|^2_{L^2(\Pi)}} \\ &\geq \inf_{\substack{S \subset D_0(r_\varepsilon^-) \\ \dim S = j}} \sup_{\substack{u \in S \\ u \neq 0}} \frac{(1-b\varepsilon)h_\varepsilon(\Psi u, \Psi u)}{\|\Psi u\|^2_{L^2(0,a)\oplus\mathcal{H}}} \\ &\geq (1-b\varepsilon) \inf_{\substack{S \subset D(h_\varepsilon) \\ \dim S = j}} \sup_{\substack{v \in S \\ v \neq 0}} \frac{h_\varepsilon(v,v)}{\|v\|^2_{L^2(0,a)\oplus\mathcal{H}}} = (1-b\varepsilon)\Lambda_j(H_\varepsilon).\end{aligned} \tag{44}$$

By Lemma 7, for some $K_0 > 0$ and for all sufficiently small $\varepsilon > 0$ we have

$$E_j(M_{(1-b\varepsilon)^2\varepsilon,a}) \leq E_j(A_{(1-b\varepsilon)^2\varepsilon}) + K_0 = -\frac{N_\omega^2}{(2j+n-2)^2(1-b\varepsilon)^4\varepsilon^2} + K_0 < 0,$$

hence, $\Lambda_j(H_\varepsilon) = \Lambda_j(M_{(1-b\varepsilon)^2\varepsilon,a} \oplus 0) = E_j(M_{(1-b\varepsilon)^2\varepsilon,a})$, and it follows by (44) that $\Lambda_j(T_\varepsilon) + c' \geq (1-b\varepsilon)E_j(M_{(1-b\varepsilon)^2\varepsilon,a})$. By (13) we have $E_j(M_{(1-b\varepsilon)^2\varepsilon,a}) \geq E_j(A_{(1-b\varepsilon)^2\varepsilon})$, therefore,

$$\begin{aligned}\Lambda_j(T_\varepsilon) &\geq (1-b\varepsilon)E_j(A_{(1-b\varepsilon)^2\varepsilon}) - c' \\ &= -\frac{N_\omega^2}{(2j+n-2)^2(1-b\varepsilon)^3\varepsilon^2} - c' \geq -\frac{N_\omega^2}{(2j+n-2)^2\varepsilon^2} - \frac{k'}{\varepsilon}\end{aligned}$$

for a suitably chosen $k' > 0$ and all sufficiently small $\varepsilon > 0$. □

By combining Corollary 14 and Lemma 15 we obtain the main result of the section:

Proposition 16. *For any $j \in \mathbb{N}$ there holds*

$$\Lambda_j(T_\varepsilon) = -\frac{N_\omega^2}{(2j+n-2)^2\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right) \text{ as } \varepsilon \to 0^+.$$

4 End of proof of Theorem 1

Note that the right-hand side of the asymptotics in Proposition 16 corresponds to the sought asymptotics for $E_j(Q_\varepsilon)$ in Theorem 1. In order to conclude the proof of Theorem 1 it remains to show that the eigenvalues of Q_ε and T_ε with the same numbers are closed to each other. This will be done in several steps.

Lemma 17. *For any $j \in \mathbb{N}$ and $\varepsilon > 0$ there holds $\Lambda_j(Q_\varepsilon) \leq \Lambda_j(T_\varepsilon)$.*

Proof. Let $J : L^2(V_\varepsilon) \to L^2(\Omega_\varepsilon)$ be the operator of extension by zero, then J is a linear isometry with $JD(t_\varepsilon) \subset D(q_\varepsilon)$ and with $q_\varepsilon(Ju, Ju) = t_\varepsilon(u,u)$ for all $u \in D(t_\varepsilon)$, and the result follows directly by the min-max principle. □

Recall that the subspaces $C_I^\infty(\overline{\Omega}_\varepsilon)$ were defined in (6). For $b>0$ denote

$$\begin{aligned}
\widetilde{V}_{\varepsilon,b} &:= \Omega_\varepsilon \cap \{x_1 > b\} \equiv \Big\{(x_1,x') \in (b,\infty)\times\mathbb{R}^n : x' \in \varepsilon x_1 \omega\Big\} \subset \mathbb{R}^{n+1},\\
\partial_0 \widetilde{V}_{\varepsilon,b} &:= \partial\Omega_\varepsilon \cap \{x_1 > b\} \equiv \Big\{(x_1,x') \in (b,\infty)\times\mathbb{R}^n : x' \in \varepsilon x_1 \partial\omega\Big\} \subset \partial\widetilde{V}_{\varepsilon,b},\\
\widehat{H}_0^1(\widetilde{V}_{\varepsilon,b}) &= \text{the closure of } C^\infty_{(b,\infty)}(\overline{\Omega}_\varepsilon) \text{ in } H^1(\widetilde{V}_{\varepsilon,b})
\end{aligned}$$

and let $\widetilde{T}_{\varepsilon,b}$ be the self-adjoint operator in $L^2(\widetilde{V}_{\varepsilon,b})$ defined by its symmetric bilinear form

$$\widetilde{t}_{\varepsilon,b}(u,u) = \int_{\widetilde{V}_{\varepsilon,b}} |\nabla u|^2 \,\mathrm{d}x - \int_{\partial_0\widetilde{V}_{\varepsilon,b}} u^2 \,\mathrm{d}\sigma, \quad D(\widetilde{t}_{\varepsilon,b}) = \widehat{H}_0^1(\widetilde{V}_{\varepsilon,b}).$$

Lemma 18. *For any $\varepsilon_0>0$ and $b>0$ there exists $c>0$ such that*

$$\inf\operatorname{spec}\widetilde{T}_{\varepsilon,b} \geq -\frac{c}{\varepsilon} \text{ for all } \varepsilon\in(0,\varepsilon_0).$$

Proof. Let $u \in C^\infty_{(b,\infty)}(\overline{\Omega}_\varepsilon)$, then due to Lemma 8 one has

$$\begin{aligned}
\widetilde{t}_{\varepsilon,b}(u,u) \geq \int_{\widetilde{V}_{\varepsilon,b}} |\partial_{x_1} u|^2 \,\mathrm{d}x + \int_b^\infty \Big\{ \int_{\varepsilon x_1\omega} |\nabla_{x'} u(x_1,x')|^2 \,\mathrm{d}x' \\
- (\varepsilon x_1)^{n-1}\sqrt{1+R^2\varepsilon^2} \int_{\partial\omega} u(x_1,\varepsilon x_1 t)^2 \,\mathrm{d}\tau(t)\Big\} \,\mathrm{d}x_1. \quad (45)
\end{aligned}$$

We have

$$\begin{aligned}
\int_{\varepsilon x_1\omega} |\nabla_{x'} u(x_1,x')|^2 \,\mathrm{d}x' &= (\varepsilon x_1)^n \int_\omega |(\nabla_{x'} u)(x_1,\varepsilon x_1 t)|^2 \,\mathrm{d}t\\
&= (\varepsilon x_1)^{n-2} \int_\omega |\nabla_t u(x_1,\varepsilon x_1 t)|^2 \,\mathrm{d}t
\end{aligned}$$

and then

$$\begin{aligned}
&\int_{\varepsilon x_1\omega} |\nabla_{x'} u(x_1,x')|^2 \,\mathrm{d}x' - (\varepsilon x_1)^{n-1}\sqrt{1+R^2\varepsilon^2} \int_{\partial\omega} u(x_1,\varepsilon x_1 t)^2 \,\mathrm{d}\tau(t)\\
&\quad = (\varepsilon x_1)^{n-2}\Big[\int_\omega |\nabla_t u(x_1,\varepsilon x_1 t)|^2 \,\mathrm{d}t - \varepsilon x_1\sqrt{1+R^2\varepsilon^2}\int_{\partial\omega} u(x_1,\varepsilon x_1 t)^2 \,\mathrm{d}\tau(t)\Big]\\
&\quad = (\varepsilon x_1)^{n-2} b_{\varepsilon x_1\sqrt{1+R^2\varepsilon^2}}\Big(u(x_1,\varepsilon x_1\cdot), u(x_1,\varepsilon x_1\cdot)\Big)\\
&\quad \geq (\varepsilon x_1)^{n-2} E_1(B_{\varepsilon x_1\sqrt{1+R^2\varepsilon^2}}) \int_\omega u(x_1,\varepsilon x_1 t)^2 \,\mathrm{d}t\\
&\quad = \frac{1}{(\varepsilon x_1)^2} E_1(B_{\varepsilon x_1\sqrt{1+R^2\varepsilon^2}}) \int_{\varepsilon x_1\omega} u(x_1,x')^2 \,\mathrm{d}x'.
\end{aligned}$$

The substitution into (45) gives

$$\begin{aligned}
\widetilde{t}_{\varepsilon,b}(u,u) &\geq \int_b^\infty \frac{E_1(B_{\varepsilon x_1\sqrt{1+R^2\varepsilon^2}})}{(\varepsilon x_1)^2} \int_{\varepsilon x_1\omega} u^2 \,\mathrm{d}x' \,\mathrm{d}x_1\\
&\geq \inf_{x_1>b} \frac{E_1(B_{\varepsilon x_1\sqrt{1+R^2\varepsilon^2}})}{(\varepsilon x_1)^2} \int_b^\infty \int_{\varepsilon x_1\omega} u^2 \,\mathrm{d}x' \,\mathrm{d}x_1 \qquad (46)\\
&\equiv \inf_{x_1>b} \frac{E_1(B_{\varepsilon x_1\sqrt{1+R^2\varepsilon^2}})}{(\varepsilon x_1)^2} \|u\|^2_{L^2(\widetilde{V}_{\varepsilon,b})}.
\end{aligned}$$

By Lemma 6(c) there exists $c_0 > 0$ such that $E_1(B_r) \geq -N_\omega r - c_0 r^2$ for all $r > 0$. Hence, for any $x_1 > b$ we have

$$\begin{aligned}\frac{E_1(B_{\varepsilon x_1\sqrt{1+R^2\varepsilon^2}})}{(\varepsilon x_1)^2} &\geq \frac{-N_\omega \varepsilon x_1\sqrt{1+R^2\varepsilon^2} - c_0\varepsilon^2 x_1^2(1+R^2\varepsilon^2)}{(\varepsilon x_1)^2}\\ &\equiv -\frac{N_\omega\sqrt{1+R^2\varepsilon^2}}{\varepsilon x_1} - c_0(1+R^2\varepsilon^2)\\ &\geq -\frac{N_\omega\sqrt{1+R^2\varepsilon_0^2}}{b\varepsilon} - c_0(1+R^2\varepsilon_0^2) \geq -\frac{c}{\varepsilon}\end{aligned}$$

with $c := \Big(N_\omega\sqrt{1+R^2\varepsilon_0^2} + b\varepsilon_0 c_0(1+R^2\varepsilon_0^2)\Big)/b$, and the substitution into (46) gives the result. □

Lemma 19. *Let $j \in \mathbb{N}$. Then there exist $K > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there holds $\Lambda_j(Q_\varepsilon) \geq \Lambda_j(T_\varepsilon) - K$.*

Proof. The argument uses the same idea as in Lemma 7. Let $\chi_1, \chi_2 \in C^\infty(0,\infty)$ with $0 \leq \chi_1, \chi_2 \leq 1$ and $\chi_1^2 + \chi_2^2 = 1$, such that $\chi_1(s) = 0$ for $s \geq \frac{3a}{4}$ and $\chi_2(s) = 0$ for $s \leq \frac{a}{2}$. We set $K := \|\chi_1'\|_\infty^2 + \|\chi_2'\|_\infty^2$ and define functions $\rho_j : (x_1, x') \mapsto \chi_j(x_1)$, then $\rho_1^2 + \rho_2^2 = 1$ and $\|\nabla\rho_1\|_\infty^2 + \|\nabla\rho_2\|_\infty^2 = K$. It is convenient to denote $b := \frac{a}{4}$. For any $u \in C^\infty_{(0,\infty)}(\overline{\Omega}_\varepsilon)$ one has

$$\begin{aligned}\int_{\Omega_\varepsilon} |\nabla u|^2 \,\mathrm{d}x &= \int_{\Omega_\varepsilon} |\nabla(\rho_1 u)|^2 \,\mathrm{d}x + \int_{\Omega_\varepsilon} |\nabla(\rho_2 u)|^2 \,\mathrm{d}x - \int_{\Omega_\varepsilon} u^2(|\nabla\rho_1|^2 + |\nabla\rho_2|^2)\,\mathrm{d}x\\ &\geq \int_{\Omega_\varepsilon} |\nabla(\rho_1 u)|^2 \,\mathrm{d}x + \int_{\Omega_\varepsilon} |\nabla(\rho_2 u)|^2 \,\mathrm{d}x - K\int_{\Omega_\varepsilon} u^2\,\mathrm{d}x.\end{aligned}$$

As $\rho_1 u$ vanishes for $x_1 > \frac{3a}{4}$ and $\rho_2 u$ vanishes for $x_1 < \frac{a}{2}$, one can rewrite the last inequality as

$$\int_{\Omega_\varepsilon} |\nabla u|^2 \,\mathrm{d}x + K\int_{\Omega_\varepsilon} u^2\,\mathrm{d}x \geq \int_{V_\varepsilon} |\nabla(\rho_1 u)|^2 \,\mathrm{d}x + \int_{\widetilde{V}_{\varepsilon,b}} |\nabla(\rho_2 u)|^2 \,\mathrm{d}x.$$

Also remark that $\rho_1 u \in \widehat{H}_0^1(V_\varepsilon)$ and $\rho_2 u \in \widehat{H}_0^1(\widetilde{V}_{\varepsilon,b})$, and

$$\begin{aligned}\int_{\partial\Omega_\varepsilon} |u|^2\,\mathrm{d}\sigma &= \int_{\partial\Omega_\varepsilon} |\rho_1 u|^2\,\mathrm{d}\sigma + \int_{\partial\Omega_\varepsilon} |\rho_2 u|^2\,\mathrm{d}\sigma = \int_{\partial_0 V_\varepsilon} |\rho_1 u|^2\,\mathrm{d}\sigma + \int_{\partial_0\widetilde{V}_{\varepsilon,b}} |\rho_2 u|^2\,\mathrm{d}\sigma,\\ \|u\|^2_{L^2(\Omega_\varepsilon)} &= \int_{\Omega_\varepsilon} |\rho_1 u|^2\,\mathrm{d}x + \int_{\Omega_\varepsilon} |\rho_2 u|^2\,\mathrm{d}x\\ &= \int_{V_\varepsilon} |\rho_1 u|^2\,\mathrm{d}x + \int_{\widetilde{V}_{\varepsilon,b}} |\rho_2 u|^2\,\mathrm{d}x = \|\rho_1 u\|^2_{L^2(V_\varepsilon)} + \|\rho_2 u\|^2_{L^2(\widetilde{V}_{\varepsilon,b})}.\end{aligned}$$

Substituting these computations into the expression for $q_\varepsilon(u,u)$ we obtain

$$q_\varepsilon(u,u) + K\|u\|^2_{L^2(\Omega_\varepsilon)} \geq t_\varepsilon(\rho_1 u, \rho_1 u) + \widetilde{t}_{\varepsilon,b}(\rho_2 u, \rho_2 u) \text{ for any } u \in C^\infty_{(0,\infty)}(\overline{\Omega}_\varepsilon),$$

and it follows, using the min-max principle, that for any $j \in \mathbb{N}$ there holds

$$\begin{aligned}
\Lambda_j(Q_\varepsilon) + K &= \inf_{\substack{S \subset C^\infty_{(0,\infty)}(\overline{\Omega}_\varepsilon) \\ \dim S = j}} \sup_{\substack{u \in S \\ u \neq 0}} \frac{q_\varepsilon(u,u) + K\|u\|^2_{L^2(\Omega_\varepsilon)}}{\|u\|^2_{L^2(\Omega_\varepsilon)}} \\
&\geq \inf_{\substack{S \subset C^\infty_{(0,\infty)}(\overline{\Omega}_\varepsilon) \\ \dim S = j}} \sup_{\substack{u \in S \\ u \neq 0}} \frac{t_\varepsilon(\rho_1 u, \rho_1 u) + \widetilde{t}_{\varepsilon,b}(\rho_2 u, \rho_2 u)}{\|\rho_1 u\|^2_{L^2(V_\varepsilon)} + \|\rho_2 u\|^2_{L^2(\widetilde{V}_{\varepsilon,b})}} \qquad (47) \\
&\geq \inf_{\substack{S \subset D(t_\varepsilon) \oplus D(\widetilde{t}_{\varepsilon,b}) \\ \dim S = j}} \sup_{\substack{(u_1,u_2) \in S \\ (u_1,u_2) \neq 0}} \frac{t_\varepsilon(u_1,u_1) + \widetilde{t}_{\varepsilon,b}(u_2,u_2)}{\|u_1\|^2_{L^2(V_\varepsilon)} + \|u_2\|^2_{L^2(\widetilde{V}_{\varepsilon,b})}} = \Lambda_j(T_\varepsilon \oplus \widetilde{T}_{\varepsilon,b}).
\end{aligned}$$

Now let $j \in \mathbb{N}$ be fixed. As $\varepsilon \to 0^+$, by Proposition 16 we have $\Lambda_j(T_\varepsilon) \sim -c\varepsilon^{-2}$ with some $c > 0$, and by Lemma 18 we have the bound $\inf \operatorname{spec} \widetilde{T}_{\varepsilon,b} \geq -\widetilde{c}\varepsilon^{-1}$ with some $\widetilde{c} > 0$. So for all sufficiently small $\varepsilon > 0$ one has $\Lambda_j(T_\varepsilon) < \inf \operatorname{spec} \widetilde{T}_{\varepsilon,b}$, which implies $\Lambda_j(T_\varepsilon \oplus \widetilde{T}_{\varepsilon,b}) = \Lambda_j(T_\varepsilon)$. The substitution into (47) finishes the proof. □

The following assertion together with the asymptotics of $\Lambda_j(T_\varepsilon)$ from Proposition 16 completes our proof of Theorem 1:

Proposition 20. *Let $j \in \mathbb{N}$ be fixed, then:*

- *one can find some $\varepsilon_j > 0$ such that Q_ε has at least j discrete eigenvalues below $\inf \operatorname{spec}_{\mathrm{ess}} Q_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_j)$,*
- *there holds $E_j(Q_\varepsilon) = \Lambda_j(T_\varepsilon) + O(1)$ as $\varepsilon \to 0^+$.*

Proof. Let us fix $j \in \mathbb{N}$. By combining the upper bound of Lemma 17 and the lower bound of Lemma 19 we obtain $\Lambda_j(Q_\varepsilon) = \Lambda_j(T_\varepsilon) + O(1)$. By Proposition 16 we have $\Lambda_{j+1}(T_\varepsilon) - \Lambda_j(T_\varepsilon) \to +\infty$ as $\varepsilon \to 0^+$. It follows that there exists $\varepsilon_j > 0$ such that $\Lambda_j(Q_\varepsilon) < \Lambda_{j+1}(Q_\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_j)$, and then $E_j(Q_\varepsilon) = \Lambda_j(Q_\varepsilon)$ for the same ε due to the min-max principle. □

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