Polyhomogeneous functions, regularized integrals, push-forward theorem etc.*

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Contents

1 Polyhomogeneous functions

Roughly speaking, a function u of a single variable $x > 0$ is polyhomogeneous if it has an asymptotic expansion as $x \to 0$ of the form

$$u(x) \sim \sum_{z,k} a_{z,k}\, x^z \log^k x, \quad a_{z,k} \in \mathbb{C} \tag{1}$$

Here $z \in \mathbb{C}$, $k \in \mathbb{N}_0$, and for each z only finitely many $a_{z,k}$ are non-zero. We write $\log^k x = (\log x)^k$. Functions of this sort arise in various ways:

- As solutions of differential equations.
- As results of integrating smooth functions (see the push-forward theorem).

*Preliminary notes for Seminar Singuläre Analysis, SS 2013 and WS 2016/17 and SS 2019

- etc.

We will do the following:

1. Make precise the meaning of the asymptotic expansion; this includes fixing the sets of (z, k) which may occur in the expansion (index sets). We will also want to be able to 'differentiate the asymptotics'[1], so we make this requirement part of the definition of (1).

2. Allow dependence of u and the $a_{z,k}$ on additional variables (parameters); geometrically this means considering functions on a half space rather than a half line

3. Characterize polyhomogeneity in terms of differential equations

4. Extend this to asymptotics in terms of several variables approaching a limit. That is, consider functions on quadrants/octants etc. instead of a half line; or more generally quadrants/octants... times Euclidean spaces if parameters are present.

5. Think about coordinate invariance. This leads to generalization to manifolds with corners.

1.1 Preliminaries, polyhomogeneous functions on the half line

First note that (all asymptotics are meant as $x \to 0$)

$$x^z \log^k x = o(x^{z'} \log^{k'} x) \text{ iff } \begin{cases} \operatorname{Re} z > \operatorname{Re} z' & \text{or} \\ \operatorname{Re} z = \operatorname{Re} z', k < k' \end{cases}$$

For example, the sequence of functions

$$\log^2 x,\ \log x,\ x^i \log x,\ 1,\ x \log x,\ x,\ x^2,\ x^\pi \log^{10} x$$

is decreasing with respect to the order $g \succ f :\iff f = o(g)$.

Therefore, the asymptotic series in (1) makes sense if the sum runs over $(z, k) \in E$ where E satisfies condition (a) in the following definition.

Definition 1.1. *An **index set** is a subset $E \subset \mathbb{C} \times \mathbb{N}_0$ satisfying*

(a) For each $s \in \mathbb{R}$ the set
$$E_{\leq s} := \{(z, k) \in E : \operatorname{Re} z \leq s\}$$
is finite.

(b) $(z, k) \in E$, $0 \leq l \leq k \Rightarrow (z, l) \in E$.

*E is a C^∞**-index set** if in addition*

(c) $(z, k) \in E \Rightarrow (z + 1, k)$.

We also denote
$$\inf E := \min\{\operatorname{Re} z : (z, k) \in E \text{ for some } k\}$$

[1] since we want to use asymptotics in solving differential equations

Condition (b) means that with any $\log x$ power also all the lesser powers may appear (with the same x^z). We will see presently why this is useful. Condition (c) will be important when considering invariance under coordinate changes.

In the sequel the differential expression (operator) $x\partial_x := x\frac{\partial}{\partial x}$ will occur frequently. One reason for this is that it behaves very nicely (much better than ∂_x) with functions of the form $x^z \log^k x$:

$$x\partial_x(x^z) = z\, x^z, \quad x\partial_x(\log^k x) = k \log^{k-1} x$$

and therefore

$$x\partial_x(x^z \log^k x) = z\, x^z \log^k x + k\, x^z \log^{k-1} x \tag{2}$$

so the vector space spanned by $x^z \log^j x$, $j = 0, \dots, k$, is invariant under the operator $x\partial_x$, for any $z \in \mathbb{C}$ and $k \in \mathbb{N}_0$. This would be false for the operator ∂_x.

In the following definitions we write

$$\mathbb{R}_+ = [0, \infty), \quad \mathrm{int}(\mathbb{R}_+) = (0, \infty)$$

Definition 1.2. *Let E be an index set. A* ***polyhomogeneous function on $\mathbb{R}_+$ with index set*** *E is a smooth function $u : \mathrm{int}(\mathbb{R}_+) \to \mathbb{C}$ for which there are $a_{z,k} \in \mathbb{C}$, $(z, k) \in E$, so that for all $j \in \mathbb{N}_0$ and $s \in \mathbb{R}$ we have*

$$(x\partial_x)^j \left(u(x) - \sum_{(z,k)\in E_{\le s}} a_{z,k}\, x^z \log^k x \right) = O(x^s) \tag{3}$$

In this case we write

$$u(x) \sim \sum_{z,k} a_{z,k}\, x^z \log^k x$$

The space of polyhomogeneous functions on $\mathbb{R}_+$ with index set E is denoted

$$\mathcal{A}^E(\mathbb{R}_+)$$

Here we use the

Convention: All O estimates are to be understood as locally uniform on the spaces in question.

That is, $f(x) = O(x^s)$ means that for any compact set $K \subset \mathbb{R}_+$ there is a constant C so that $|f(x)| \le Cx^s$ for all $x \in K$ at which f is defined. The main point is that we have a statement about behavior of $f(x)$ as $x \to 0$ (since K may contain zero), but none about its behavior as $x \to \infty$.

Remarks 1.3 (on Definition 1.2)**.**

1. *$u(x)$ is only defined for $x > 0$, but we say that u is polyhomogeneous on $[0, \infty)$ since there is a condition on the behavior of u in arbitrarily small pointed neighborhoods of 0.*

2. *If we required (3) only for $j = 0$ then we would get the standard notion of asymptotic series (no derivatives).*

   Requiring (3) for $j = 1$ means that 'the asymptotics can be formally differentiated': Condition (3) for $\partial_x u$ and the formally differentiated series $\sum_{z,k} a_{z,k}\, \partial_x(x^z \log^k x)$, with $j = 0$, is

$$\partial_x u(x) - \sum_{(z,k)\in E_{\le s}} a_{z,k}\, \partial_x(x^z \log^k x) = O(x^{s-1})$$

since x^{s-1} is the last power of x subtracted on the left. Multiplying this by x one sees that this is equivalent to condition (3) for u and $j = 1$.

This is another (if not the main) reason why $x\partial_x$ appears all over the place.

3. *We would obtain the same space of functions if we required*

$$(x\partial_x)^j \left(u(x) - \sum_{\substack{(z,k)\in E \\ \operatorname{Re} z < s}} a_{z,k}\, x^z \log^k x \right) = O(x^{s-\varepsilon}) \tag{3'}$$

*for all j, s and $\varepsilon > 0$. This is the definition used in [**?**].*

It is slightly less obvious that one also gets the same space of functions if one replaces the right hand side of (3) or (3′) by $O(x^{s-N})$ and requires that there exists *an N so that the estimate holds for all j, s. (exercise)*

Proposition 1.4. *Let E be an index set. Then $\mathcal{A}^E(\mathbb{R}_+)$ is a vector space, and is mapped by $x\partial_x$ to itself.*

This is obvious from (2). Note that for the last statement one needs condition (b) in Definition 1.1 and all j in Definition 1.2.

Definition 1.2 may be made more digestible by introducing some notation. We will do this in the more general case of a half space in Lemma 1.9.

Examples 1.5.

1. *A function is smooth on $\mathbb{R}_+$ if and only if it is polyhomogeneous with index set $\mathbb{N}_0 \times \{0\}$. One implication in this equivalence follows directly from Taylor's theorem, applied to the function and its derivatives. The other direction is an exercise.* (refer to manifolds with corners chapter)
2. *Clearly, x^{-2}, x^e, $\log x$ are polyhomogeneous on $\mathbb{R}_+$ with suitable index sets.*
3. *The function $\sin \frac{1}{x}$ is not polyhomogeneous on $\mathbb{R}_+$ for any index set: Its fast oscillation as $x \to 0$ cannot be modelled using functions of the form $x^z \log^k x$. If we used the Taylor series $\sin \frac{1}{x} = \frac{1}{x} - \frac{1}{6}\frac{1}{x^3} \pm \ldots$ then arbitrarily large negative powers of x would appear. This is not allowed for an index set.*

1.2 Polyhomogeneous functions on the half space

We now consider functions on the half space

$$H^n := \mathbb{R}_+ \times \mathbb{R}^{n-1}$$

It is standard to denote the variables

$$x \in \mathbb{R}_+, \quad y = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$$

The definition of polyhomogeneity extends in a straightforward way, where we want to assume smooth dependence on y and also the possibility to differentiate the asymptotic series in y.

Definition 1.6. *Let E be an index set. A **polyhomogeneous function on H^n with index set** E is a smooth function $u : \operatorname{int}(H^n) \to \mathbb{C}$ for which there are $a_{z,k} \in C^\infty(\mathbb{R}^{n-1})$, $(z,k) \in E$, so that for all $j \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^{n-1}$ and $s \in \mathbb{R}$ we have*

$$(x\partial_x)^j \partial_y^\alpha \left(u(x,y) - \sum_{(z,k)\in E_{\leq s}} a_{z,k}(y)\, x^z \log^k x \right) = O(x^s) \tag{4}$$

In this case we write

$$u(x,y) \sim \sum_{(z,k)\in E} a_{z,k}(y)\, x^z (\log x)^k$$

The space of polyhomogeneous functions on H^n with index set E is denoted

$$\mathcal{A}^E(H^n)$$

Recall the convention that O estimates are meant to be uniform on compact subsets. Here this means compact subsets of H^n.

The following definitions are designed to focus attention on various aspects of this definition. First, it is useful to give a name to combinations of derivatives as they occur in (4).

Definition 1.7. *A **b-differential operator** on H^n is an operator of the form*

$$\sum_{j,\alpha} b_{j,\alpha}(x,y)(x\partial_x)^j \partial_y^\alpha$$

where $b_{j,\alpha} \in C^\infty(H^n)$ for all $j \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^{n-1}$ and only finitely many terms of the sum are non-zero. The space of b-differential operators on H^n is denoted $\mathrm{Diff}_b^*(H^n)$.

As usual the **order** of $P \in \mathrm{Diff}_b^*(H^n)$ is defined as the largest $j + |\alpha|$ for which $b_{j,\alpha}$ is not identially zero, and by $\mathrm{Diff}_b^m(H^n)$ we denote the set of b-operators of order at most m. Clearly this is a vector space, and the composition of two b-operators is a b-operator.

We now introduce spaces in which the remainders – the expressions in parantheses in (4) – lie.

Definition 1.8. *For $s \in \mathbb{R}$ let*

$$\mathcal{A}^s(H^n) = \{u \in C^\infty(\mathrm{int}(H^n)) : \ Pu = O(x^s) \text{ for all } P \in \mathrm{Diff}_b^*(H^n)\}$$

Equivalently, $(x\partial_x)^j \partial_y^\alpha u = O(x^s)$ for all j, α. Functions in $\mathcal{A}^s(H^n)$ are sometimes called **conormal** with respect to the boundary ∂H^n. The definition of polyhomogeneity translates directly as:

Lemma 1.9. *A function $u \in C^\infty(\mathrm{int}(H^n))$ is polyhomogeneous, $u(x,y) \sim \sum_{(z,k)\in E} a_{z,k}(y)\, x^z (\log x)^k$ if and only if for each $s \in \mathbb{R}$ we can write*

$$u = \sum_{(z,k)\in E_{\leq s}} a_{z,k}\, x^z \log^k x \ + \ r_s, \quad r_s \in \mathcal{A}^s(H^n) \tag{5}$$

Remark 1.10. *Equivalently, instead of $r_s \in \mathcal{A}^s(H^n)$ $\forall s$ we may require the seemingly weaker condition that $r_s \in \mathcal{A}^{s'}(H^n)$ $\forall s$, for some s' which tends to infinity as $s \to \infty$. (For example $s' = s - 1$ or $s' = s/2$.) (Proof as exercise.)*

As before, we have

Proposition 1.11. *Let E be an index set. Then $\mathcal{A}^E(H^n)$ and $\mathcal{A}^s(H^n)$ are vector spaces, and are mapped by $\mathrm{Diff}_b^*(H^n)$ to themselves.*

1.3 Characterization by differential operators

In order to prove some basic properties of polyhomogeneous functions, it is useful to characterize them in a different way.

The starting point is the observation that $(x\partial_x - z)x^z = 0$, and more generally

$$(x\partial_x - z)x^z \log^k x = kx^z \log^{k-1} x \tag{6}$$

which implies $(x\partial_x - z)^{k+1}x^z \log^k x = 0$. A neat way to understand this is by noticing that $x\partial_x - z = x^z(x\partial_x)x^{-z}$ (conjugation of $x\partial_x$ by the operator of multiplication by x^z), which reduces the claims to the case $z = 0$.

More precisely and more generally, we have for any finite subset $S \subset \mathbb{C}$ and numbers $p_z \in \mathbb{N}_0$

$$\ker \prod_{z\in S}(x\partial_x - z)^{p_z+1} = \{\sum_{z\in S}\sum_{k=0}^{p_z} a_{z,k}\, x^z \log^k x,\ a_{z,k} \in \mathbb{C}\} \tag{7}$$

as functions on $\mathrm{int}(\mathbb{R}_+)$: Clearly the functions on the right are in the kernel, and then the equality follows from a dimension argument, using that the set of functions $x^z \log^k x$, $(z, k) \in E$, is linearly independent.

The right side of (7) is simply a 'piece' of a polyhomogeneous expansion! This makes the following theorem plausible.

Theorem 1.12. *For each $s \in \mathbb{R}$ define the differential operator*

$$B_{E,s} = \prod_{(z,k)\in E_{\le s}} (x\partial_x - z)$$

Then

$$\mathcal{A}^E(H^n) = \{u \in C^\infty(\mathrm{int}(H^n)) :\ B_{E,s}u \in \mathcal{A}^s(H^n) \text{ for all } s \in \mathbb{R}\} \tag{8}$$

In this characterization of polyhomogeneity the coefficients $a_{z,k}$ do not appear explicitly!

Note that the factor $x\partial_x - z$ appears $p + 1$ times in $B_{E,s}$ if $p = \max\{k : (z, k) \in E\}$. This implies

$$\{v \in C^\infty(\mathrm{int}(H^n) :\ B_{E,s}v = 0\} = \{\sum_{(z,k)\in E_{\le s}} a_{z,k}(y)x^z \log^k x :\ a_{z,k} \in C^\infty(\mathbb{R}^{n-1})\} \tag{9}$$

by the remarks before the theorem (applied for any fixed y; the $a_{z,k}$ must be smooth in y by smoothness of u).

Proof. First, let $u \in \mathcal{A}^E(H^n)$. For any $s \in \mathbb{R}$, write u as in (5). Then (9) implies $B_{E,s}u = B_{E,s}r_s$, and by Proposition 1.11 this lies in $\mathcal{A}^s(H^n)$ since $B_{E,s} \in \mathrm{Diff}^*_b(H^n)$.

We have proved the inclusion '$\subset$' of (8). To prove the converse, we use the following lemma.

Lemma 1.13. *Let $s \in \mathbb{R}$, $z \in \mathbb{C}$.*

1. *$u \in \mathcal{A}^s \Rightarrow x^z u \in \mathcal{A}^{s+\mathrm{Re}\, z}$*
2. *If $u \in \mathcal{A}^s$ then there is $w \in \mathcal{A}^s$ such that $(x\partial_x - z)w = u$.*

Proof. Note that $x\partial_x x^z = x^z(x\partial_x + z)$. Applying this repeatedly, we see that for any $P \in \mathrm{Diff}_b^*(H^n)$ there is $P' \in \mathrm{Diff}_b^*(H^n)$ satisfying $Px^z = x^z P'$. This implies (1) since $|x^z| = x^{\mathrm{Re}\, z}$. Using conjugation by x^z and (1) we may assume $z = 0$ in (2). Set

$$w(x,y) = \begin{cases} \int_0^x u(t,y)\frac{dt}{t} & \text{if } s > 0 \\ \int_1^x u(t,y)\frac{dt}{t} & \text{if } s \leq 0 \end{cases}$$

Then $x\partial_x w = u$, and $u = O(x^s)$ implies $w = O(x^s)$ if $s > 0$, and $w = O(1 + x^s) = O(x^s)$ if $s \leq 0$, and similarly for the $\partial_y^\alpha w$ estimates. The estimates of $(x\partial_x)^j \partial_y^\alpha w$ for $j \geq 1$ follow from $x\partial_x w = u$ and the estimates for u. □

To finish the proof of Theorem 1.12 assume $u \in C^\infty(\mathrm{int}(H^n))$ satisfies $B_{E,s}u \in \mathcal{A}^s(H^n)$ for all $s \in \mathbb{R}$. Fix s and let $\tilde{u} = B_{E,s}u$. Applying the lemma iteratively find $w \in \mathcal{A}^s$ with $\tilde{u} = B_{E,s}w$. Then $B_{E,s}(u - w) = 0$, hence by (9) there are $a_{z,k} \in C^\infty(\mathbb{R}^{n-1})$ for $\mathrm{Re}\, z \leq s$ so that $u - w = \sum_{(z,k)\in E_{\leq s}} a_{z,k}\, x^z \log^k x$. It is easy to check that when the same procedure is done for $s' > s$, producing coefficients $a'_{z,k}$, then one must have $a'_{z,k} = a_{z,k}$ for $\mathrm{Re}\, z \leq s$. It follows that $u \sim \sum_{(z,k)\in E} a_{z,k}(y)\, x^z \log^k x$, so $u \in \mathcal{A}^E(H^n)$. □

1.4 Polyhomogeneous functions on a quadrant

We will first discuss polyhomogeneous functions on the simplest manifold with corners, the quadrant $\mathbb{R}^2_+$. This will guide us how to proceed for general manifolds with corners.

What's the idea? Polyhomogeneity of a smooth function u on $\mathrm{int}(\mathbb{R}^2_+)$ should involve three things:

1. $u(x,y)$ should be polyhomogeneous as $x \to 0$, smoothly in $y > 0$.
2. $u(x,y)$ should be polyhomogeneous as $y \to 0$, smoothly in $x > 0$.
3. These expansions should be uniform, in a suitable sense, at the corner, i.e. for $x \to 0$ *and* $y \to 0$.

The smooth dependence in 1. and 2. should be as in the case of the half space, but it is less clear how to make 3. precise. There are different ways to do this. First, we should fix index sets for both side faces.

Definition 1.14. *Let M be a manifold with corners. An **index family** $\mathcal{E}$ for M is an assignment of an index set $\mathcal{E}(H)$ to each boundary hypersurface H of M.*

For $M = \mathbb{R}^2_+$ we denote an index family simply by (E, F), where E is considered as index set for $\{x = 0\}$ and F is an index set for $\{y = 0\}$.[2]

Next, we extend the definitions of b-differential operators and conormal spaces to this case.

Definition 1.15. *A **b-differential operator** on $\mathbb{R}^2_+$ is an operator of the form*

$$\sum_{j,l} b_{j,l}(x,y)(x\partial_x)^j(y\partial_y)^l$$

where $b_{j,l} \in C^\infty(\mathbb{R}^2_+)$ for all $j, l \in \mathbb{N}_0$ and only finitely many terms of the sum are non-zero. The space of b-differential operators on $\mathbb{R}^2_+$ is denoted $\mathrm{Diff}_b^*(\mathbb{R}^2_+)$.

[2] This is opposite to the notation used in [**?**].

Definition 1.16. *For $s, t \in \mathbb{R}$ let*

$$\mathcal{A}^{(s,t)}(\mathbb{R}^2_+) = \{u \in C^\infty(\mathrm{int}(\mathbb{R}^2_+)) : Pu = O(x^s y^t) \text{ for all } P \in \mathrm{Diff}^*_b(\mathbb{R}^2_+)\}$$

Here the local uniformity implicit in the O is for compact subsets of $\mathbb{R}^2_+$.

Definition 1.17. *Let (E, F) be an index family for $\mathbb{R}^2_+$. A* ***polyhomogeneous function on $\mathbb{R}^2_+$ with index family*** *(E, F) is a smooth function $u : \mathrm{int}(\mathbb{R}^2_+) \to \mathbb{C}$ for which there are*

$$a_{z,k} \in \mathcal{A}^F(\mathbb{R}_+),\ (z, k) \in E \text{ and } b_{w,l} \in \mathcal{A}^E(\mathbb{R}_+),\ (w, l) \in F$$

and $N \in \mathbb{R}$ so that for all $s \in \mathbb{R}$ we have

$$u = \sum_{(z,k)\in E_{\leq s}} a_{z,k}(y)x^z \log^k x + r_s, \quad r_s \in \mathcal{A}^{(s,-N)}(\mathbb{R}^2_+) \tag{10}$$

$$u = \sum_{(w,l)\in F_{\leq s}} b_{w,l}(x)y^w \log^l y + r'_s, \quad r'_s \in \mathcal{A}^{(-N,s)}(\mathbb{R}^2_+) \tag{11}$$

The $-N$ should be thought of as any number smaller than $\inf E$ and $\inf F$. It is needed since both u and the sum on the right in (10) will behave like $y^{\inf E}$ times logarithms, and similarly for (11).

Examples 1.18.

1. *u is smooth on $\mathbb{R}^2_+$ if and only if it is polyhomogeneous with index sets $E = F = \mathbb{N}_0 \times \{0\}$.*

2. *The function $u(x, y) = \sqrt{x^2 + y^2}$ is smooth on $\mathbb{R}^2 \setminus \{(0, 0)\}$, so it has polyhomogeneous expansions in the interior of each boundary hypersurface. However, u is not polyhomogeneous on $\mathbb{R}^2_+$. To see this, we find the expansion at the face $x = 0$ by writing, for $y > 0$,*

$$\sqrt{x^2 + y^2} = y\sqrt{1 + (x/y)^2} = y \sum_0^\infty c_i (\frac{x}{y})^{2i} \tag{12}$$

$$= y + \frac{1}{2}\frac{x^2}{y} - \frac{1}{8}\frac{x^4}{y^3} + \dots \tag{13}$$

with the Taylor series $\sqrt{1 + t} = \sum_0^\infty c_i t^i = 1 + t/2 - t^2/8 + \dots$ (for $|t| < 1$). Thus, in the expansion $u(x, y) \sim \sum_{i=0}^\infty a_{2i}(y)x^{2i}$ the coefficients are $a_{2i}(y) = c_i y^{1-2i}$. Although each a_{2i} is polyhomogeneous as $y \to 0$, there is no index set F so that each a_{2i} has the same index set F. Therefore, polyhomogeneity at the corner fails.

Remark 1.19. *Equations (10) and (11) imply that the coefficient functions $a_{z,k}$, $b_{w,l}$ must satisfy compatibility conditions at the corner: When we write*

$$a_{z,k}(y) \sim \sum_{(w,l)\in F} c_{z,k,w,l}\, y^w \log^l y, \quad b_{2,l}(x) \sim \sum_{(z,k)\in E} c'_{z,k,w,l}\, x^z \log^k x$$

then necessarily $c_{z,k,w,l} = c'_{z,k,w,l}$ for all z, k, w, l. (Exercise!)

Remark 1.20. *It may seem more natural to require that the remainder r_s in (10) vanish to order s in x and be polyhomogeneous in y: $r_s \in \mathcal{A}^{s,F}$, and similarly for (11).*

In this approach we would need to define the space $\mathcal{A}^{s,F}$ first. This can be done, and then this stronger condition follows from the one given above. (exercise) (CHECK DETAILS) This will then imply (expand each $a_{z,k}$ and r_s in y, up to order t) that for all $s,t \in \mathbb{R}$ one has

$$u = u_{12} + u_1 + u_2 + r$$

$$u_{12}(x,y) = \sum_{\operatorname{Re} z \le s, \operatorname{Re} w \le t} c_{c,k,w,l}\, x^z \log^k x\, y^w \log^l y$$

$$u_1(x,y) = \sum_{\operatorname{Re} z \le s} r_{z,k}(y) x^z \log^k x\,,\ \ u_2(x,y) = \sum_{\operatorname{Re} w \le t} r'_{w,l}(x) y^w \log^l y$$

$$r_{z,k} \in \mathcal{A}^t(\mathbb{R}_+),\ r'_{w,l} \in \mathcal{A}^s(\mathbb{R}_+),\ r \in \mathcal{A}^{(s,t)}(\mathbb{R}^2_+)$$

where always $(z,k) \in E$ and $(w,l) \in F$. The index in u_{12} etc. indicates at which face the function does not vanish to higher order. So u_{12} is a corner term, u_1 is an x-axis-term and u_2 is a y-axis-term.

This representation, and its straightforward generalization to higher dimensions, allows for a simple direct proof of the push-forward theorem (as indicated in footnote 17 of (BBC) for smooth functions).

Exercise 1.21. *Check that the y^{-N} (resp. x^{-N}) bound for the remainders in* (10), (11) *is essential. That is, if we just require that u has asymptotics $u(x,y) \sim \sum_{(z,k)\in E} a_{z,k}(y)\, x^z (\log x)^k$, locally uniformly in $y \in (0,\infty)$ analogous to Definition 1.6, and similarly with x, y interchanged, then we cannot conclude that u is polyhomogeneous.*

Hint: Choose $\varphi \in C^\infty((0,\infty))$ having polyhomogeneous expansions a 0 *and at ∞, and consider $u(x,y) = \varphi(\frac{y}{x})$.*

Again we have a characterization of polyhomogeneity by differential operators, which avoids explicit appearance of the coefficient functions.

Theorem 1.22. *For each $s \in \mathbb{R}$ define the differential operators*

$$B^x_{E,s} = \prod_{(z,k)\in E_{\le s}} (x\partial_x - z)$$

$$B^y_{F,s} = \prod_{(w,l)\in F_{\le s}} (y\partial_y - w)$$

Then $u \in \mathcal{A}^{(E,F)}(\mathbb{R}^2_+)$ iff u is smooth in the interior and there is $N \in \mathbb{R}$ so that for all $s \in \mathbb{R}$

$$B^x_{E,s} u \in \mathcal{A}^{(s,-N)}(\mathbb{R}^2_+), \quad B^y_{F,s} u \in \mathcal{A}^{(-N,s)}(\mathbb{R}^2_+) \tag{14}$$

See [?].

Proof. Analogous to the proof of Theorem 1.12. □

(14) implies that

$$B^x_{E,s} B^y_{F,s} u \in \mathcal{A}^{(t,t)}(\mathbb{R}^2_+), \quad t = \frac{s-N}{2}$$

since the spaces $\mathcal{A}^{(s,t)}(\mathbb{R}^2_+)$ are invariant under b-differential operators and $\mathcal{A}^{(s,-N)}(\mathbb{R}^2_+) \cap \mathcal{A}^{-N.s}(\mathbb{R}^2_+) \subset \mathcal{A}^{(t,t)}(\mathbb{R}^2_+)$. The proof of this inclusion is left as an exercise.

TODO (not essential but interesting): (10) (with quantifiers) implies (11).

1.5 Polyhomogeneous functions on general model spaces

Next we consider the spaces

$$\mathbb{R}^n_k := \mathbb{R}^k_+ \times \mathbb{R}^{n-k}$$

which are the local models for general manifolds with corners. The extension of the previous discussion to this case is straightforward: The $\mathbb{R}^{n-k}$ variables are treated like (smooth) parameters as in the case of a halfspace, and having a codimension k corner is analogous to a codimension 2 corner. This should really be worked out by the reader as an exercise, but we provide the main steps.

By convention, we denote the coordinates as $x_1, \dots, x_k \in \mathbb{R}_+$, $y_1, \dots, y_{n-k} \in \mathbb{R}$ and also $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_{n-k})$. Then **b-differential operators** are, by definition, operators of the form

$$\sum_{\alpha \in \mathbb{N}_0^k, \beta \in \mathbb{N}_0^{n-k}} b_{\alpha,\beta}(x,y)(x_1\partial_{x_1})^{\alpha_1} \dots (x_k \partial_{x_k})^{\alpha_k} \partial_y^\beta \tag{15}$$

with all $b_{\alpha,\beta}$ smooth on $\mathbb{R}^n_k$. For $s_1, \dots, s_k \in \mathbb{R}$ we define the **conormal spaces**

$$\mathcal{A}^{(s_1,\dots,s_k)}(\mathbb{R}^n_k) = \{u \in C^\infty(\mathrm{int}(\mathbb{R}^n_k)) : \ Pu = O(x_1^{s_1} \dots x_k^{s_k}) \text{ for all } P \in \mathrm{Diff}^*_b(\mathbb{R}^n_k)\}$$

As always, the O is to be understood as being locally uniform on $\mathbb{R}^n_k$.

An index family for $\mathbb{R}^n_k$ is given by k index sets $E_1, \dots, E_k$, with E_j associated to the boundary hypersurface $\{x_j = 0\}$ for each j. We want to define the space

$$\mathcal{A}^{(E_1,\dots,E_k)}(\mathbb{R}^n_k)$$

of polyhomogeneous functions on $\mathbb{R}^n_k$ with index family $(E_1, \dots, E_k)$. The definition of polyhomogeneity works by induction over k. Suppose we have defined polyhomogeneous functions on spaces $\mathbb{R}^n_{k-1}$ for any n, then a **polyhomogeneous function on $\mathbb{R}^n_k$ with index family** $(E_1, \dots, E_k)$ is a smooth function $u : \mathrm{int}(\mathbb{R}^n_k) \to \mathbb{C}$ so that there are $N \in \mathbb{R}$ and functions for each $j = 1, \dots, k$

$$a^{(j)}_{z,k} \in \mathcal{A}^{(E_1,\dots,\widehat{E_j},\dots,E_k)}(\mathbb{R}^{n-1}_{k-1}), \quad (z,k) \in E_j$$

with the hat denoting omission, such that for all $s \in \mathbb{R}$ and for each j we have

$$u = \sum_{(z,k)\in(E_j)_{\le s}} a^{(j)}_{z,k}(x_{\neq j}, y)\, x_j^z \log^k x_j + r^{(j)}_s, \quad r^{(j)}_s \in \mathcal{A}^{(-N,\dots,s,\dots,-N)}(\mathbb{R}^n_k) \tag{16}$$

where $x_{\neq j} = (x_1, \dots, \widehat{x_j}, \dots, x_k)$ and the s is at the jth spot.

As for $\mathbb{R}^2_+$ this implies (multiple) compatibility relations for the expansion coefficients for different j, and also we have a characterization by differential operators analogous to Theorem 1.22.

Note that if we allow the functions $a^{(j)}_{z,k}$ in (16) to depend on *all* variables (including x_j) then by expanding them in Taylor series around $x_j = 0$ we get an expansion of the type (16), but with additional terms involving $x_j^{z+m} \log^k x_j$ for $m \in \mathbb{N}_0$. This is where part (c) of Definition 1.1 matters, and we get:

Lemma 1.23. *Suppose each E_j is a C^∞ index set. Then $\mathcal{A}^{(E_1,\dots,E_k)}(\mathbb{R}^n_k)$ is equal to the space of functions having expansions as in* (16) *but with the $a^{(j)}_{z,k}$ depending on all variables x, y.*

Note that allowing these more general coefficients has the disadvantage that they are not uniquely determined by u (while those in (16) are). However, the lemma will be needed when discussing coordinate invariance.

1.6 The invariant perspective: Manifolds with corners.

The only additional issue which arises when we consider manifolds with corners is invariance under coordinate changes. We first discuss this for $\mathbb{R}_+$. The standard coordinate x on $\mathbb{R}_+$ is a boundary defining function for the boundary hypersurface $\{0\}$. A general boundary defining function is a function on $\mathbb{R}_+$ which vanishes at 0, has non-vanishing derivative there, and is positive on $(0, \infty)$. By Taylor's theorem, it can be written as

$$x' = x\rho(x), \quad \rho \in C^\infty(\mathbb{R}_+),\ \rho > 0 \text{ on } \mathbb{R}_+$$

Then

$$(x')^z \log^k x' = x^z \rho(x)^z (\log x + \log \rho(x))^k$$

Now ρ is smooth and positive, hence $\log \rho$ and ρ^z are smooth. Expanding these functions in Taylor series around $x = 0$ and multiplying out, we see that

$$(x')^z \log^k x' \sim \sum_{m=0}^{\infty} \sum_{l=0}^{k} \gamma_{m,l}\, x^{z+m} \log^l x$$

for certain coefficients $\gamma_{m,l}$. The fact that also the powers x^{z+m} appear on the right is another reason, besides Lemma 1.23, for the condition (c) in Definition (1.1).

Now consider coordinate changes on $\mathbb{R}^n_k$. A general boundary defining function for the boundary hypersurface $\{x_j = 0\}$ is of the form

$$x'_j = x_j \rho(x, y), \quad \rho \in C^\infty(\mathbb{R}^n_k), \rho > 0$$

A simple inductive argument together with Lemma 1.23 shows that:

Proposition 1.24. *Let $\mathcal{E}$ be an index family for $\mathbb{R}^n_k$. If each index set in $\mathcal{E}$ is a C^∞-index set then the space $\mathcal{A}^{\mathcal{E}}(\mathbb{R}^n_k)$ is invariant under changes of coordinates.*

Also, it is clear that the definitions of these spaces are local in the sense that $u \in \mathcal{A}^{\mathcal{E}}(\mathbb{R}^n_k)$ if and only if $\rho u \in \mathcal{A}^{\mathcal{E}}(\mathbb{R}^n_k)$ for every $\rho \in C^\infty(\mathbb{R}^n_k)$. Therefore, if we define, for any open subset $U \subset \mathbb{R}^n_k$,

$$\mathcal{A}^{\mathcal{E}}(U) = \{u \in C^\infty(U \cap \mathrm{int}(\mathbb{R}^n_k)) : \rho u \in \mathcal{A}^{\mathcal{E}}(\mathbb{R}^n_k) \text{ for all } \rho \in C_0^\infty(U)\}$$

then this is compatible with the previous definition in case $U = \mathbb{R}^n_k$. This allows us to define:

Definition 1.25. *Let M be a manifold with corners and $\mathcal{E}$ a C^∞ index set for M. A **polyhomogeneous function on** M **with index family** $\mathcal{E}$ is a smooth function $u : \mathrm{int}(M) \to \mathbb{C}$ which is polyhomogeneous with corresponding index family in any local chart.*

Explicitly, this means that for any chart $\varphi : \tilde{U} \to U$, $\tilde{U} \subset \mathbb{R}^n_k$ open, we have $\varphi^* u \in \mathcal{A}^{\mathcal{E}'}(\tilde{U})$, where $\mathcal{E}'(\phi^{-1}(H \cap U)) := \mathcal{E}(H)$ for every boundary hypersurface H of M which intersects U.

We can also generalize the remaining discussion to the manifold case. We start by reformulating the definition of b-differential operators on $\mathbb{R}^n_k$, (15), in an invariant way. First, consider first order operators annihilating constants, i.e. vector fields. Note that any smooth vector field on $\mathbb{R}^n_k$ can be written

$$\sum_{j=1}^{k} a_j \partial_{x_j} + \sum_{l=1}^{n-k} b_l \partial_{y_l}$$

with smooth functions a_j, b_l. Such a vector field is tangent to the boundary hypersurface $\{x_j = 0\}$ if and only if $a_j = 0$ at $x_j = 0$, which is equivalent to $a_j = x_j a'_j$ for some smooth function a'_j. This shows that

$$\begin{gathered}\{\text{smooth vector fields on } \mathbb{R}^n_k \text{ which are tangent to all boundary hypersurfaces}\} \\ = \operatorname{span}_{C^\infty(\mathbb{R}^n_k)}\{x_1\partial_{x_1}, \dots, x_k\partial_{x_k}, \partial_{y_1}, \dots, \partial_{y_{n-k}}\} := \\ \{\sum_{j=1}^{k} a_j\, x_j\partial_{x_j} + \sum_{l=1}^{n-k} b_l\partial_{y_l},\ a_j, b_l \in C^\infty(\mathbb{R}^n_k)\,\forall j,l\}\end{gathered}$$

Then clearly, for $m \in \mathbb{N}_0$,

$$\mathrm{Diff}_b^m(\mathbb{R}^n_k) = \{a + \sum_{r=1}^{m} \sum_{V_1,\dots,V_r \in \mathcal{V}_b(\mathbb{R}^n_k)} V_1 \dots V_r :\ a \in C^\infty(\mathbb{R}^n_k)\}$$

This generalizes naturally to manifolds:

Definition 1.26. *Let M be a manifold with corners. Then define*

$$\mathcal{V}_b(M) = \{\textit{smooth vector fields on } M \textit{ which are tangent to all boundary hypersurfaces}\}$$

and for $m \in \mathbb{N}_0$

$$\mathrm{Diff}_b^m(M) = \{a + \sum_{r=1}^{m} \sum_{V_1,\dots,V_r \in \mathcal{V}_b(M)} V_1 \dots V_r :\ a \in C^\infty(M)\}$$

This leads directly to conormal spaces:

Definition 1.27. *Let M be a manifold with corners. A* ***weight family*** *for M is a map $\mathcal{M}_1(M) \to \mathbb{R}$, where $\mathcal{M}_1(M)$ is the set of boundary hypersurfaces of M.*

For a set of boundary defining functions ρ_H for each $H \in \mathcal{M}_1(M)$ and for a weight family $\mathfrak{s}$ define $\rho^\mathfrak{s} = \prod_{H\in\mathcal{M}_1(M)} \rho_H^{\mathfrak{s}(H)}$. Finally, define the conormal space

$$\mathcal{A}^\mathfrak{s}(M) = \{u \in C^\infty(\mathrm{int}(M)) :\ Pu = O(\rho^\mathfrak{s}) \textit{ for all } P \in \mathrm{Diff}_b^*(M)\}$$

As always, the O is understood locally uniformly on M. This is a reasonable definition since clearly the set of functions which are $O(\rho^\mathfrak{s})$ is independent of the choice of the ρ_H. Also, we see that in the case of $M = \mathbb{R}^n_k$ we get back the previous definition.

Now we have generalizations of all previous results:

- $C^\infty(M) = \mathcal{A}^{\mathcal{E}_0}(M)$ where $\mathcal{E}_0(H) = \mathbb{N}_0 \times \{0\}$ for all H.
- $\mathcal{A}^\mathfrak{s}(M)$ and $\mathcal{A}^\mathcal{E}(M)$ are vector spaces and invariant under $\mathrm{Diff}_b^*(M)$.
- There is also a characterization of $\mathcal{A}^\mathcal{E}(M)$ using vector fields. This is a little subtle since we need to think carefully about an invariant generalization of the vector fields $x_i\partial_{x_i}$ in Theorem 1.22. The main observation is that, for $M = \mathbb{R}^n_k$, the vector field $x_i\partial_{x_i}$ ($i \in \{1, \dots, k\}$) turns under any coordinate change into a vector field of the form

$$x_i\partial_{x_i} + x_i V,\ V \in \mathcal{V}_b(\mathbb{R}^n_k) \tag{17}$$

and that the space of these vector fields is invariant under coordinate changes. We call a b-vector field on M **radial** with respect to the boundary hypersurface H if it has the form (17) in any coordinate system, with x_i defining H.

Also, it is easily checked that (14) remains true if $x\partial_x$, $y\partial_y$ are replaced by any radial vector fields for the respective boundary hypersurfaces. In light of this, the following theorem is natural, and we leave the details of the proof to the reader.

Theorem 1.28. *Let M be a manifold with corners and $\mathcal{E}$ a C^∞ index family for M. For each $H \in \mathcal{M}_1(M)$ choose a radial vector field V_H and define*

$$B^H_{\mathcal{E},s} = \prod_{(z,k)\in\mathcal{E}(H)_{\leq s}} (V_H - z)$$

Then a smooth function u on $\operatorname{int}(M)$ is in $\mathcal{A}^{\mathcal{E}}(M)$ if and only if there is $N \in \mathbb{R}$ so that for all $s \in \mathbb{R}$ and all $H \in \mathcal{M}_1(M)$

$$B^H_{\mathcal{E},s} u \in \mathcal{A}^{\mathfrak{s}_H}(M), \quad \mathfrak{s}_H(H') := \begin{cases} s & H' = H \\ -N & H' \neq H \end{cases}$$

2 Push-forward theorem and coefficient formulas (preliminary)

A classic problem which occurs in many applications is to find the asymptotic expansion of an integral

$$I(t) = \int u(x,t)\, dx$$

under various assumptions on u. Often, such an expansion can be found, even if the integral cannot be evaluated explicitly. Or at least the qualitative behavior of I can be determined.

A simple case is that u is smooth in both variables x, t, and compactly supported. Then I is also smooth. However, it is not obvious at all what happens for an integral of the form

$$I(t) = \int u(x, \frac{t}{x})\, dx$$

where, say, $u \in C_0^\infty(\mathbb{R}_2^+)$. Clearly, I is smooth for $t > 0$, but is it also smooth at $t = 0$? It turns out that the answer is NO: I has a polyhomogeneous expansion, but there will also be terms involving $\log t$. This is one reason why we need to allow logarithms in the definition of polyhomogeneity! More generally, if u is polyhomogeneous then $I(t)$ will also be, but with higher log powers.

Integrals of this kind occur often in singular analysis. They are a special case of a very general operation called push-forward under a b-fibration.

We will analyze the integral above in Section 2.2 and the general push-forward theorem in **??**. In the formulas for the coefficients of the asymptotics of I we will encounter 'integrals of non-integrable functions', so we discuss these first.

2.1 Regularized integrals

Often the need arises to integrate functions that are not integrable. For example, this will occur in the solution of the problem mentioned above; also, such integrals arise frequently in global analysis and in quantum field theory. How can such a notion be reasonably defined?

2.1.1 Introducing the ideas in the case of smooth functions

We formulate the problem more precisely, in a special case. The discussion is summarized in Theorem 2.3 and Definition 2.4, skip there for quick reading.

Consider the expression

$$I_0(u) = \int_0^\infty u(x)\,\frac{dx}{x}\,, \quad u \in \mathcal{F} := C_0^\infty(\mathbb{R}_+). \tag{18}$$

This is not defined if $u(0) \neq 0$. However, if $u(0) = 0$ then $u(x) = O(x)$ near $x = 0$, so the integral converges. By definition, a *regularization* of I_0 is a linear extension of I_0 from the subspace $\mathcal{F}_0 := \{u \in \mathcal{F} : u(0) = 0\}$ to all of $\mathcal{F}$. How can we find a regularization?

First, how many regularizations can there be? Since $\mathcal{F}_0$ is a hyperplane in $\mathcal{F}$, defined by the vanishing of the linear functional $\delta : u \mapsto u(0)$, we know:

> There is a one-dimensional space of regularizations, and if I_1, I_2 are regularizations then $I_1 - I_2 = c\delta$ for some $c \in \mathbb{C}$.

Here are three ideas how one could explicitly define regularizations:

1. *Subtract* $u(0)$*:* If $u(0)$ is a problem, subtract it!

   Replacing $u(x)$ by $u(x) - u(0)$ in the integral (18) won't help since then we get non-integrability for x near ∞, therefore we choose a cutoff function ρ and define

   $$I_\rho(u) = \int_0^\infty [u(x) - u(0)\rho(x)]\,\frac{dx}{x} \quad \text{for } \rho \in C_0^\infty(\mathbb{R}_+), \rho(0) = 1\,. \tag{19}$$

   This is clearly well-defined (convergent), linear, and for $u(0) = 0$ yields $I_0(u)$. So I_ρ is a regularization of I_0. How does it depend on the choice of ρ? Clearly

   $$I_\rho - I_{\tilde\rho} = c\delta\,, \quad c = \int_0^\infty (\tilde\rho(x) - \rho(x))\,\frac{dx}{x}\,. \tag{20}$$

   Note that the integral defining c exists although $\int_0^\infty \rho(x)\,\frac{dx}{x}$ doesn't.

   Note that I_ρ is defined if ρ is C^1 near $x = 0$ and compactly supported, for example if ρ is the characteristic function of $[0, 1]$.

2. *Cut off near zero:* If there is a problem at $x = 0$, cut it off!

   Consider $f(\varepsilon) = \int_\varepsilon^\infty u(x)\,\frac{dx}{x}$ for $\varepsilon > 0$. If $u(0) = 0$ then $f(\varepsilon) \to I_0(u)$ as $\varepsilon \to 0$. How does $f(\varepsilon)$ behave for general u? It diverges logarithmically, more precisely:

   Lemma 2.1. *As* $\varepsilon \to 0$*,*

   $$\int_\varepsilon^\infty u(x)\,\frac{dx}{x} = a\log\varepsilon^{-1} + b + O(\varepsilon)$$

   where $a = u(0)$*,* $b = I_\chi(u)$*, with* $\chi = \chi_{[0,1]}$ *the characteristic function of* $[0, 1]$.

   Therefore, taking the constant term in the asymptotics of the function $\varepsilon \mapsto \int_\varepsilon^\infty u(x)\,\frac{dx}{x}$ as $\varepsilon \to 0$ defines a regularization of I_0, which coincides with $I_{\chi_{[0,1]}}$.

   The point is that this is defined for all $u \in C_0^\infty(\mathbb{R}_+)$, and for $u(0) = 0$ yields the standard integral I_0.

Proof. Replacing ε by 0 in $\int_\varepsilon^\infty [u(x) - u(0)\chi(x)] \frac{dx}{x}$ introduces an error of $O(\varepsilon)$, so

$$\begin{aligned}\int_\varepsilon^\infty u(x)\,\frac{dx}{x} &= \int_\varepsilon^\infty [u(x) - u(0)\chi(x)]\,\frac{dx}{x} + \int_\varepsilon^\infty u(0)\chi(x)\,\frac{dx}{x}\\ &= O(\varepsilon) + \int_0^\infty [u(x) - u(0)\chi(x)]\,\frac{dx}{x} - u(0)\log\varepsilon\end{aligned}$$

□

3. *Meromorphic extension:* If integrating $\frac{u(x)}{x}$ is a problem, integrate $x^s\frac{u(x)}{x}$ instead!
   If $u(0) = 0$ then the function

$$F(s) = \int_0^\infty u(x)\,x^s\,\frac{dx}{x}$$

   is defined and holomorphic for complex numbers s near 0, and $F(0) = I_0(u)$. If $u \in C_0^\infty(\mathbb{R}_+)$ then $F(s)$ is still well-defined if $\operatorname{Re} s > 0$, and more precisely:

   Lemma 2.2. *Let $u \in C_0^\infty(\mathbb{R}_+)$. Then the function F has a meromorphic extension from* $\operatorname{Re} s > 0$ *to a neighborhood of $s = 0$, with at most a simple pole at $s = 0$.*

   The Laurent expansion of F around $s = 0$ is

$$F(s) = \frac{a}{s} + b + O(s)$$

   where $a = u(0)$ and $b = I_\chi(u)$ with $\chi = \chi_{[0,1]}$.

   Therefore, taking the constant term in the Laurent series of F defines a regularization of I_0, which coincides with $I_{\chi_{[0,1]}}$.

   Proof. If $\operatorname{Re} s > 0$ then we may integrate by parts, with vanishing boundary terms, and obtain from $x^{s-1} = \frac{1}{s}\frac{d}{dx}x^s$

$$F(s) = \frac{G(s)}{s}, \quad G(s) = -\int_0^\infty u'(x)x^s\,dx\,.$$

   The latter integral converges for $\operatorname{Re} s > -1$ and defines a holomorphic function $G(s)$ there. So F extends meromorphically to this set with at most a simple pole at $s = 0$. The residue is $G(0) = -\int_0^\infty u'(x)\,dx = u(0)$, and the constant term is

$$\begin{aligned}G'(0) &= -\int_0^\infty u'(x)\log x\,dx = \lim_{\varepsilon\to 0} -\int_\varepsilon^\infty u'(x)\log x\,dx\\ &= \lim_{\varepsilon\to 0} -u(x)\log x|_\varepsilon^\infty + \int_\varepsilon^\infty u(x)\,\frac{dx}{x}\\ &= \lim_{\varepsilon\to 0} u(0)\log\varepsilon + O(\varepsilon\log\varepsilon) + \int_\varepsilon^\infty u(x)\,\frac{dx}{x}\\ &= I_\chi(u)\end{aligned} \tag{21}$$

   by Lemma 2.1. □

We summarize the discussion. Note that we used very few assumptions on u: it needs to be C^1 in a neighborhood of $x = 0$ and sufficiently integrable away from zero, i.e. on any interval $[\varepsilon, \infty)$ with $\varepsilon > 0$.

Theorem 2.3. *Let*

$$\mathcal{F} = \{u : \mathbb{R}_+ \to \mathbb{C} : u \text{ is } C^1 \text{ near } x = 0 \text{ and in } L^1(x^a \frac{dx}{x}) \text{ away from } x = 0, \text{ for some } a > 0\}.$$

The functional

$$u \mapsto \int_0^\infty u(x)\,\frac{dx}{x}$$

defined on $\{u \in \mathcal{F} :\ u(0) = 0\}$ *has a one-dimensional space of linear extensions to* $\mathcal{F}$*. Each of the following definitions defines the same extension.*

(a) $I(u) = \displaystyle\int_0^\infty \left[u(x) - u(0)\chi_{[0,1]}(x)\right]\,\frac{dx}{x}$

(b) $I(u)$ *is the constant term in the asymptotic expansion of*

$$\varepsilon \mapsto \int_\varepsilon^\infty u(x)\,\frac{dx}{x} \qquad \text{as } \varepsilon \to 0.$$

(c) $I(u)$ *is the constant term in the Laurent expansion of*

$$s \mapsto \int_0^\infty u(x)\,x^s\,\frac{dx}{x} \qquad \text{around } s = 0.$$

(d) I *is essentially the distributional derivative of the function*

$$\log_+ : x \mapsto \begin{cases} \log x & \text{if } x > 0 \\ 0 & \text{else.} \end{cases}$$

More precisely, $I(u_{|\mathbb{R}_+}) = \tilde{I}(u)$ *if* $\tilde{I}$ *is this distributional derivative and* $u \in C_0^\infty(\mathbb{R})$*.*

Proof. The equivalence of (a), (b), (c) follows from Lemmas 2.1 and 2.2.[3] The extra integrability away from $x = 0$ (i.e. $a > 0$ in the definition of $\mathcal{F}$) is needed for the meromorphic extension in (c) only. The equivalence of (a) and (d) follows from the calculation (21). □

Definition 2.4. *The number* $I(u)$ *given in Theorem 2.3 is called the* ***regularized integral*** *of* $u\,\frac{dx}{x}$*. It is denoted by*

$$⨍_0^\infty u(x)\,\frac{dx}{x}\,.$$

Remark 2.5. *Let us consider different aspects of the characterizations (a)-(d).*

Locality *Defining a regularization is a local problem at* $x = 0$*. This is best captured by (b).*

Naturality *(a) shows most clearly which choice is involved in defining the regularization, see also* (20)*. The choice in (b), (c) is in the choice of coordinate* x*. We discuss this in Subsection 2.1.4.*

[3] Exercise: Check that the proofs work on the bigger function space $\mathcal{F}$.

Distribution perspective *For* $\operatorname{Re} s > 0$ *define the locally integrable function* x_+^{s-1} *on* $\mathbb{R}$ *by*[4]

$$x_+^{s-1} = \begin{cases} x^{s-1} & \text{if } x > 0 \\ 0 & \text{else.} \end{cases}$$

This defines a holomorphic map $f : \{\operatorname{Re} s > 0\} \to \mathcal{D}'(\mathbb{R})$, $s \mapsto x_+^{s-1}$.[5] *It has a meromorphic continuation to* $s \in \mathbb{C}$, *defined as follows: For* $\operatorname{Re} s > 0$ *we have* $x_+^{s-1} = \frac{1}{s}(x_+^s)'$, *i.e.* $f(s) = \frac{1}{s} Df(s+1)$ *where* $D = \frac{d}{dx}$. *We use this relation to define* $f(s)$ *for* $\operatorname{Re} s > -1$, *then for* $\operatorname{Re} s > -2$ *etc.*

For example, we obtain the Laurent expansion of f *around* $s = 0$ *as follows: We have* $f(1) = H$ *and* $f'(1) = \frac{d}{ds}_{|s=1} x_+^{s-1} = \log_+ x$, *so* f *has Taylor expansion* $f(s+1) = H + s\log_+ + O(s^2)$, *hence*

$$f(s) = \frac{1}{s} Df(s+1) = \frac{1}{s}\delta + D\log_+ + O(s)\,.$$

This proves (again) the equivalence of parts (c) and (d) in the theorem.

Examples 2.6. *1.* $\displaystyle\int_0^\infty \chi_{[0,c]}(x)\,\frac{dx}{x} = \log c$ *for* $c > 0$.

2. $\displaystyle\int_0^\infty e^{-x}\,\frac{dx}{x} = \int_0^\infty e^{-x}\log x\,dx = -\gamma$ *where* $\gamma = 0.577\ldots$ *is the Euler-Mascheroni constant. Note that in this case the function in Theorem 2.3(c) is the Gamma function.*

2.1.2 The Mellin transform

The function in Theorem 2.3(c) is called the Mellin transform of u. It is a useful tool for generalizing the previous discussion to polyhomogeneous functions and for proving the quantitative Push-Forward Theorem, see ...

Definition 2.7. *If* $u : (0,\infty) \to \mathbb{C}$ *is locally integrable then the* ***Mellin transform*** *of* u *is the function* $\mathcal{M}u$ *defined by*

$$(\mathcal{M}u)(s) = \int_0^\infty u(x)\,x^s\,\frac{dx}{x}$$

for all $s \in \mathbb{C}$ *for which the integral is defined.*[6]

For example, if u has compact support in $(0,\infty)$ then $\mathcal{M}u$ is defined and holomorphic on $\mathbb{C}$. More generally, the following is obvious:

Lemma 2.8. *Let* $N \in \mathbb{R}$. *Suppose*

$$u \in L^1_{loc}(0,\infty) \text{ has bounded support and } u(x) = O(x^N) \tag{22}$$

Then $\mathcal{M}u$ *is defined and holomorphic on* $\operatorname{Re} s > -N$.

[4]Writing $s-1$ is more natural in our context than writing s. It would be even more natural to speak of the density $x^s\,\frac{dx}{x}$.

[5]Holomorphy means that for each test function $u \in C_0^\infty(\mathbb{R})$ the map $s \mapsto \int u(x)x_+^{s-1}\,dx$ is holomorphic.

[6]There are various slightly different definitions in the literature. These correspond to rotating the complex s plane. For example, Melrose in [?] defines $(\mathcal{M}u)(s) = \int_0^\infty u(x)\,x^{-is}\,\frac{dx}{x}$, which makes the Mellin transform more similar to the Fourier transform.

(The bounded support assumption may be replaced by exponential decay as $x \to \infty$. See Exercise 2.15 for more general functions.) Henceforth we assume that condition (22) is satisfied. Note that N may well be negative.

Important properties of the Mellin transform are (on the obvious domains)

$$\mathcal{M}(x\partial_x u) = -s\,\mathcal{M}u \tag{23}$$

$$\mathcal{M}(u \log x) = (\mathcal{M}u)' \tag{24}$$

$$\mathcal{M}(x^z u)(s) = \mathcal{M}u(s+z) \tag{25}$$

obtained by integration by parts, by differentiating in s and by trivial calculation, respectively. For (23) we assume that $x\partial_x u$ also satisfies (22).

The Mellin transform is closely related to the Fourier transform $(\mathcal{F}U)(\xi) = \int_{-\infty}^{\infty} e^{-it\xi} U(t)\,dt$. The substitution $x = e^t$ in the Mellin integral yields

$$\mathcal{M}u(s) = \mathcal{F}U(is) \quad \text{for} \quad U(t) = u(e^t) \tag{26}$$

The Fourier inversion formula then yields:

Theorem 2.9 (Mellin inversion formula). *Suppose u is twice differentiable on $(0,\infty)$ and u, $x\partial_x u$, $(x\partial_x)^2 u$ satisfy (22), where $N \in \mathbb{R}$. Let $a > -N$. Then $\mathcal{M}u$ is integrable over the vertical line $a + i\mathbb{R}$, and*

$$u(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} (\mathcal{M}u)(s) x^{-s}\,ds\,. \tag{27}$$

As a preparation for the proof, note that, if u is J times differentiable then

$$(x\partial_x)^j u = O(x^N),\ j \le J \Longrightarrow |(\mathcal{M}u)(s)| \le C_{\operatorname{Re} s} |s|^{-J}\,. \tag{28}$$

This follows directly from (23) applied J times, with $C_\sigma = \mathcal{M}(|(x\partial_x)^J u|)(\sigma)$.

Proof of Theorem 2.9. Integrability over the line $a + i\mathbb{R}$ follows from (28) with $J = 2$. In the case $N > 0$ and $a = 0$ (27) is simply the Fourier inversion formula: With (26) we have

$$u(e^t) = U(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathcal{F}U)(\xi) e^{it\xi}\,d\xi = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} (\mathcal{M}u)(s)(e^t)^{-s}\,ds$$

substituting $\xi = is$. The general case reduces to this by considering $v(x) = x^a u(x)$ instead. □

Example 2.10. *Let $\chi = \chi_{[0,1]}$ be the characteristic function of $[0,1]$. For $z \in \mathbb{C}$ and $k \in \mathbb{N}_0$ we have:*

$$u(x) = x^z \log^k x\, \chi(x) \Longrightarrow \mathcal{M}u(s) = \frac{c_k}{(s+z)^{k+1}},\ c_k = (-1)^k k! \tag{29}$$

for $\operatorname{Re} s > -\operatorname{Re} z$. This follows for $k = 0$ by direkt calculation and then in general by differentiating in s (or z) k times.

Note that in the example, $\mathcal{M}u$ has a meromorphic continuation to all of $\mathbb{C}$, with a pole of order $k+1$ at $s = -z$. This leads to one half of the following theorem.

Notation 2.11. *For a meromorphic function f and $s_0 \in \mathbb{C}$ denote*

$$\begin{aligned} \operatorname{ord}(f, s_0) &= \textit{order of the pole of } f \textit{ at the point } s_0 \\ \operatorname{Res}_{-k,s=s_0} f(s) &= \textit{coefficient of } (s-s_0)^{-k} \textit{ in the Laurent} \\ &\qquad \textit{expansion of } f(s) \textit{ around } s = s_0. \end{aligned}$$

For an index set $E \subset \mathbb{C} \times \mathbb{N}_0$ let

$$E_{\mathbb{C}} := \{z \in \mathbb{C} : (z,0) \in E\}$$
$$\operatorname{ord}_E(z) := \max\{k \in \mathbb{N}_0 : (z,k) \in E\} \quad \textit{for } z \in E_{\mathbb{C}}.$$

That is, $z \in E_{\mathbb{C}}$ if terms of the form x^z are permitted in an expansion of a function in $\mathcal{A}^E(\mathbb{R}_+)$, and $\operatorname{ord}_E(z)$ is the maximal k so that $x^z \log^k x$ is permitted.

Theorem 2.12 (Main theorem on the Mellin transform)**.** *Let u satisfy condition* (22) *for some $N_0 \in \mathbb{R}$. Let E be an index set. Then the following are equivalent:*

(i) u is polyhomogeneous with index set E.

(ii) $\mathcal{M}u$ satisfies:

(a) $\mathcal{M}u$ has a meromorphic continuation to $\mathbb{C}$, with poles at most at the points $-z$ having $z \in E_{\mathbb{C}}$ and of order

$$\operatorname{ord}(\mathcal{M}u, -z) \le 1 + \operatorname{ord}_E(z)$$

(b) $\mathcal{M}u$ decays rapidly in the imaginary direction:

$$\mathcal{M}u(s) = O(|\operatorname{Im} s|^{-J}) \ \textit{as} \ |\operatorname{Im} s| \to \infty, \ \textit{for all } J$$

uniformly in any strip $\{a \le \operatorname{Re} s \le b\}$.

Also, if u satisfies these conditions then

$$\operatorname{Res}_{-(k+1),s=-z}(\mathcal{M}u)(s) = (-1)^k k! \cdot \Big[\textit{coefficient of } x^z \log^k x \textit{ in } u(x) \textit{ as } x \to 0\Big] \tag{30}$$

In part (b) and in the sequel we use the notation $\mathcal{M}u$ also for the meromorphically continued function. Then equations (23)-(25) continue to hold for this continuation, by the principle of holomorphic continuation.

This characterization of polyhomogeneity in terms of meromorphy is the main use of the Mellin transform. It is a precise version of the general (rough) principles

$$\text{(lack of) decay of } u \text{ as } x \to 0 \longleftrightarrow \text{ (lack of) regularity of } \mathcal{M}u$$
$$\text{regularity of } u \longleftrightarrow \text{ decay of } \mathcal{M}u \text{ in the imaginary direction}$$

which is analogous to similar principles for the Fourier transform. (Here regularity means interior smoothness, plus the fact that the asymptotic expansion as $x \to 0$ is assumed to hold with derivatives.)

The factor i in (26) explains why we get decay in the imaginary direction for $\mathcal{M}u$.

Proof of Theorem 2.12. Choose a cut-off function $\rho \in C_0^\infty(\mathbb{R}_+)$ which equals 1 near $x = 0$. Recall that (i) means that for any $N \in \mathbb{R}$ we have

$$u = \sum_{(z,k)\in E_{\le N}} a_{z,k}\, x^z \log^k x\, \rho(x) \ + \ r_N, \qquad (x\partial_x)^j r_N = O(x^N) \text{ for all } j \tag{31}$$

where $E_{\le N} = \{(z,k) \in E : \operatorname{Re} z \le N\}$ and $a_{z,k} \in \mathbb{C}$. Note that using $\chi_{[0,1]}$ here instead of ρ we would obtain (ii)(a) immediately from (29) and Lemma 2.8. However, smoothness is destroyed by $\chi_{[0,1]}$, so we have no chance of getting (ii)(b) in this way.

Proof of (i)⇒(ii): $\mathcal{M}r_N$ is holomorphic in $\{\operatorname{Re} s > -N\}$ and satisfies the estimate of (ii)(b) in strips having $a > -N$. Since N is arbitrary, it suffices to prove (ii) for $u_{z,k}(x) := x^z \log^k x\, \rho(x)$. We have

$$\mathcal{M}u_{z,k}(s) = \mathcal{M}u_{0,k}(s+z) = (s+z)^{-(k+1)}\mathcal{M}((x\partial_x)^{k+1}u_{0,k})(s+z)$$

by (25) and (23). Now $(x\partial_x)^{k+1}u_{0,k} \in C_0^\infty((0,\infty))$ by the product rule and $(x\partial_x)^{k+1}\log^k x = 0$, so its Mellin transform is an entire function, and (28) implies that it satisfies (b). This proves (ii).

(ii)⇒(i): We want to prove (31) for any $N \in \mathbb{R}$. The main idea is to use the residue theorem to shift the contour in the Mellin inversion formula: We may assume that N is not the real part of any $z \in E_{\mathbb{C}}$. By Theorem 2.9 we have $u(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}(\mathcal{M}u)(s)x^{-s}\,ds$ for $a > -N_0$. The meromorphy of $\mathcal{M}u$ and the estimate in (ii)(b) guarantee that we can shift the contour of integration from $a+i\mathbb{R}$ to $-N+i\mathbb{R}$. Then we pick up residues at the poles of $\mathcal{M}u$ which evaluate to a sum of terms const$\cdot x^z\log^l x$ over $l \le k$ if $k+1$ is the order of the pole at $-z$. This gives (31) with

$$r_N(x) = \frac{1}{2\pi i}\int_{-N-i\infty}^{-N+i\infty}(\mathcal{M}u)(s)x^{-s}\,ds\,.$$

Now $|(x\partial_x)^j r_N(x)| \le x^N \int\limits_{-N-i\infty}^{-N+i\infty} |(\mathcal{M}u)(s)|\,|s|^j\,|ds| \le Cx^N$ by (ii)(b), so (31) and hence (i) follows.

Finally, formula (30) follows from Example 2.10.

□

Remarks 2.13.

1. *By* (30) *the* negative *powers of the Laurent expansions of $\mathcal{M}u$ at its poles are determined by the asymptotics of u. Below we will see that the constant term (at $s = 0$) defines a regularization of the integral $\int_0^\infty u(x)\,\frac{dx}{x}$.*
2. *As a special case, if $u \in C_0^\infty(\mathbb{R}_+)$ then $\mathcal{M}u$ has a meromorphic extension to $\mathbb{C}$ with simple poles at $s \in -\mathbb{N}_0$.*
3. *Theorems 2.9 and 2.12 still hold, with the same proof, if u does not have bounded support but is rapidly decaying at infinity in the sense that $\partial_x^j u(x) = O(x^{-M})$ for all j, M as $x \to \infty$.*

Example 2.14 (Gamma and zeta functions)**.** *For $u(x) = e^{-x}$ Theorem (2.12) gives the meromorphic continuation of the Gamma function $\Gamma = \mathcal{M}u$.*

For $u(x) = \frac{1}{e^x-1} = \sum_{n=1}^\infty e^{-nx}$ we get

$$(\mathcal{M}u)(s) = \sum_{n=1}^\infty(\mathcal{M}e^{-nx})(s) = \sum_{n=1}^\infty n^{-s}(\mathcal{M}e^{-x})(s) = \Gamma(s)\zeta(s)$$

where ζ is the Riemann zeta function, and this gives the meromorphic continuation of ζ, with the only pole at $s = 1$ since the other poles of $\mathcal{M}u$ cancel against those of Γ.

Exercise 2.15. *Extend the previous results to the case that u has polyhomogeneous asymptotic expansions at both $x = 0$ and $x = \infty$.*[7] *Check that $\mathcal{M}u \equiv 0$ if $u(x) = x^z\log^k x$.*

[7] Suppose the leading powers of x for $x \to 0$ and $x \to \infty$ have real parts A and $-B$, respectively. If $\operatorname{Re} A < \operatorname{Re} B$ then the integral defining $\mathcal{M}u$ converges in the strip $\operatorname{Re} A < \operatorname{Re} s < \operatorname{Re} B$. This condition can be avoided by cutting u into two pieces, supported on $[0,1]$ and $[1,\infty)$ respectively, and extending each piece meromorphically.

2.1.3 General definition and properties of the regularized integral

We will generalize the definition of $⨍_0^\infty u(x)\,\frac{dx}{x}$ to functions on $\mathbb{R}_+$ with an arbitrary polyhomogeneous expansion as $x \to 0$ and integrable at infinity. It turns out that (b) and (c) in Theorem 2.3 generalize nicely. We first show that they coincide in general. The following notation is useful.

Definition 2.16. *If $u \in \mathcal{A}^E(\mathbb{R}_+)$ for some index set E then define the* ***regularized limit***

$$\operatorname*{LIM}_{x\to 0} u(x) = \text{ the } x^0 \text{ term in the asymptotics of } u(x) \text{ as } x \to 0.$$

Thus, if $u(x) \sim \sum_{(z,k)\in E} a_{z,k}\, x^z \log^k x$ then $\mathrm{LIM}_{x\to 0}\, u(x) = a_{0,0}$, which is to be understood as zero if $(0,0) \notin E$. Clearly, $\mathrm{LIM}_{x\to 0}$ is a linear functional on $\mathcal{A}^E(\mathbb{R}_+)$ extending $\lim_{x\to 0}$.
Denote

$$\mathcal{A}^E_0(\mathbb{R}_+) = \{u \in \mathcal{A}^E(\mathbb{R}_+) :\, \operatorname{supp} u \text{ is compact}\}\,.$$

Recall that $\mathrm{Res}_{0,s=0}\, \mathcal{M}u(s)$ denotes the s^0 term in the Laurent expansion of u around $s = 0$. If $\mathcal{M}$ has not pole at $s = 0$ then this is simply $Mu(0)$.

Proposition 2.17 (and definition)**.** *Let E be an index set and let $u \in \mathcal{A}^E_0(\mathbb{R}_+)$.*

(a) For $x > 0$ let

$$v(x) = \int_x^\infty u(y)\,\frac{dy}{y}\,.$$

Then $v \in \mathcal{A}^{E'}_0(\mathbb{R}_+)$ with $E' = E \cup \{(0,0)\} \cup \{(0,k+1) :\, (0,k) \in E\}$.

(b) We have

$$\operatorname*{LIM}_{x\to 0} v(x) = \mathrm{Res}_{0,s=0}\, \mathcal{M}u(s)\,.$$

This number is called the ***regularized integral*** *of $u\,\frac{dx}{x}$ and denoted*

$$⨍_0^\infty u(x)\,\frac{dx}{x}$$

Clearly, the definition of the regularized integral via $\mathrm{LIM}_{x\to 0}$ extends to u being integrable over $[1,\infty)$, rather than having compact support.

The definition of E' is a special case of the extended union, defined in (39) below: $E' = E\overline{\cup}\{(0,0)\}$.

Proof.

(a) Clearly v has compact support. To find its asymptotics, note that $\int x^z \log^k x\,\frac{dx}{x}$ is, for $z \neq 0$, a linear combination of $x^z \log^l x$, $l = 0,\dots,k$;[8] and for $z = 0$ it equals $\frac{1}{k+1}\log^{k+1} x$ (plus a constant in both cases).

(b) From $x\partial_x v = -u$ we get

$$s\,\mathcal{M}v = \mathcal{M}u$$

from (23). Therefore, $\mathrm{Res}_{0,s=0}(\mathcal{M}u)(s) = \mathrm{Res}_{-1,s=0}(\mathcal{M}v)(s)$, which equals $\mathrm{LIM}_{x\to 0}\, v(x)$ by (30) applied with $z = k = 0$.

[8]For a simple proof, first do $k = 0$, then differentiate k times in z.

$\square$

Example 2.18. *For $z \in \mathbb{C}$, $k \in \mathbb{N}_0$ we have*

$$\fint_0^1 x^z \log^k x \, \frac{dx}{x} = \begin{cases} (-1)^k k! \dfrac{1}{z^{k+1}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \tag{32}$$

If we use the Mellin transform definition then this follows from (29). Alternatively, $\int_x^1 y^z \log^k y \, \frac{dy}{y}$ evaluates to $\left(\frac{d}{dz}\right)^k \frac{1-x^z}{z}$ if $z \neq 0$ and to $\log^{k+1} y|_x^1$ if $z = 0$. Now take the x^0 coefficient in either case.

Remark 2.19. *Clearly, $\fint_0^\infty$ defines a linear functional $\mathcal{A}_0^E(\mathbb{R}_+) \to \mathbb{C}$. Also, with $E_{>0} := \{(z,k) \in E : \operatorname{Re} z > 0\}$ we have*

$$u \in \mathcal{A}_0^{E_{>0}}(\mathbb{R}_+) \implies \fint_0^\infty u(x) \, \frac{dx}{x} = \int_0^\infty u(x) \, \frac{dx}{x} \, ,$$

so $\fint_0^\infty$ is a linear extension of the functional $\int_0^\infty$ from the subspace $\mathcal{A}_0^{E_{>0}}(\mathbb{R}_+)$ to all of $\mathcal{A}_0^E(\mathbb{R}_+)$.
What is special about this extension, why don't we choose another one? The codimension of $\mathcal{A}_0^{E_{>0}}(\mathbb{R}_+)$ in $\mathcal{A}_0^E(\mathbb{R}_+)$ is the finite number $|E \setminus E_{>0}|$, so there are many other extensions. [9]
Here are some possible answers:

- ***('Fundamental Theorem of Calculus' for the regularized integral)***
  For $u \in \mathcal{A}_0^{E_{>0}}(\mathbb{R}_+)$ we have by the fundamental theorem of calculus

  $$\int_0^\infty u(x) \, \frac{dx}{x} = 0 \iff \exists v : u = x\partial_x v \text{ and } v(0) = 0.$$

  This continues to hold for $u \in \mathcal{A}_0^E(\mathbb{R}_+)$ if we use the regularized integral on the left and replace $v(0) = 0$ by $\mathrm{LIM}_{x\to 0} v(x)$ on the right. This property characterizes the extension $\fint_0^\infty$ uniquely.
- *The regularized integral as defined above occurs as asymptotic coefficient in the coefficient formulas for the push-forward theorem, see ...*
- *Does $\fint$ have naturality properties which characterize it?*

For integrable $u \, \frac{dx}{x}$ we have $\int_0^\infty u(\lambda x) \, \frac{dx}{x} = \int_0^\infty u(x) \, \frac{dx}{x}$. This almost extends to the regularized integral, but with an additional term from the $x^0 \log^k x$ terms:

Proposition 2.20 (Change of variables for regularized integrals)**.**
Let $u \in \mathcal{A}_0^E(\mathbb{R}_+)$, $u(x) \sim \sum_{(z,k)\in E} a_{z,k} \, x^z \log^k x$. Then for $\lambda > 0$

$$\fint_0^\infty u(\lambda x) \, \frac{dx}{x} = \fint_0^\infty u(x) \, \frac{dx}{x} - \sum_{k\geq 0} \frac{\log^{k+1} \lambda}{k+1} a_{0,k} \tag{33}$$

[9] For example, for any cut-off function $\rho \in L^1_{\mathrm{comp}}(\mathbb{R}_+)$ which equals one near $x = 0$ we have an extension defined by

$$I_\rho(u) = \int_0^\infty [u(x) - u_0(x)\rho(x)] \, \frac{dx}{x}$$

where $u_0(x) = \sum_{(z,k)\in E_{\leq 0}} a_{z,k} \, x^z \log^k x$ collects the non-integrable terms of $u(x) \sim \sum_{(z,k)\in E} a_{z,k} \, x^z \log^k x$. This is analogous to (19). Note that the equivalence of Theorem 2.3(a) and (b) does *not* extend, more precisely $I_{\chi_{[0,1]}} \neq \fint_0^\infty$ unless $\operatorname{Re} z \geq 0$ for all $(z,k) \in E$, as follows from (32). Check this!

That is, each term $\log^k x$ in the asymptotics of u contributes an extra term $-\frac{\log^{k+1}\lambda}{k+1}$. Note that only $z = 0$ contributes here. For example,

$$u \in C_0^\infty(\mathbb{R}_+) \Longrightarrow \fint_0^\infty u(\lambda x)\,\frac{dx}{x} = \fint_0^\infty u(x)\,\frac{dx}{x} - u(0)\log\lambda$$

Proof. By linearity it suffices to check this for integrable $u(x)\,\frac{dx}{x}$, where it is obvious, and for $u_{z,k}(x) = x^z \log^k x\, \chi_{[0,1]}(x)$. Now $\mathcal{M}(u(\lambda x))(s) = \lambda^{-s}\mathcal{M}u(s)$ for any u, so

$$\fint_0^\infty u(\lambda x)\,\frac{dx}{x} - \fint_0^\infty u(x)\,\frac{dx}{x} = \operatorname{Res}_{0,s=0}\,(\lambda^{-s}-1)\mathcal{M}u(s)\,.$$

If $z \neq 0$ then this value is zero since $\lambda^0 = 1$ and $\mathcal{M}u_{z,k}(s)$ is holomorphic at $s = 0$. If $z = 0$ then $\mathcal{M}u_{0,k}(s) = \frac{(-1)^k k!}{s^{k+1}}$ by (29). Also, $\lambda^{-s} = e^{-s\log\lambda} = \sum_{j=0}^\infty \frac{1}{j!}(-\log\lambda)^j s^j$, so

$$\operatorname{Res}_{0,s=0}\,(\lambda^{-s}-1)\mathcal{M}u_{0,k}(s) = (-1)^k k!\frac{1}{(k+1)!}(-\log\lambda)^{k+1} = -\frac{\log^{k+1}\lambda}{k+1}\,.$$

□

Exercise 2.21. *Check that the coordinate change $y = x^a$ with $a > 0$ works 'as usual':*

$$\fint_0^\infty u(x^a)\,\frac{dx}{x} = \frac{1}{a}\fint_0^\infty u(y)\,\frac{dy}{y}\,.$$

2.1.4 Invariant perspective (unfinished...)

A compactly supported density can be integrated invariantly over a manifold. An integrable b-density can also be integrated over $\mathbb{R}_+$. However, the regularized integral is not defined invariantly, due to the extra term in Proposition 2.20.

Lemma 2.22. *The regularized integral of b-density $u(x)\,\frac{dx}{x}$ with $u \in \mathcal{A}_0^E(\mathbb{R}_+)$ is unchanged by a coordinate change $y(x) = x + O(x^N)$ if $\operatorname{Re} z > -N$ for all z with $(z,0) \in E$.*

For example, coordinate changes $y(x) = x + O(x^2)$ are permissible if only powers x^z having $\operatorname{Re} z > -1$ occur in $u(x)$.

Refinement: only integer powers in u are a problem (for E a smooth index set) (of course one could consider more general coordinate changes like $y = x^a$ for any $a > 0$, as in D. Joyce 2016, then this would not be the case).

Definition 2.23 (Boundary defining function to order N)**.**

Definition 2.24. *regularized integral of b-density in $\mathcal{A}_0^{\mathcal{E}}(M)$ on a manifold with corners M, given a choice of bdf for each bhs H of M to order N_H where $N_H > -\operatorname{Re}\mathcal{E}(H)$.*

2.2 The push-forward theorem and coefficient formula for the map $f(x,y) = xy$ (Sketch)

Goal of this section: State PFT for $f(x,y) = xy$, give four proofs:

1. Using vector fields characterization of phg

2. first two coefficients in smooth case by direct estimates

3. Using Mellin transform

4. 'Direct' (as in footnote 17 of BBC)

1. is the easiest, but does not give formulas for the coefficients. 2., 3. and 4. give formulas for the coefficients. 4. is most direct ('direct calculation', done right)

Recall that $u \in \mathcal{A}^{E,F}(\mathbb{R}^2_+)$ means that u is smooth in $(0,\infty)^2$ and has boundary expansions

$$u(x,y) \overset{x\to 0}{\sim} \sum_{(z,k)\in E} a_{z,k}(y) x^z \log^k x \tag{34}$$

$$u(x,y) \overset{y\to 0}{\sim} \sum_{(w,l)\in F} b_{w,l}(x) y^w \log^l y \tag{35}$$

where

$$a_{z,k}(y) \overset{y\to 0}{\sim} \sum_{(w,l)\in F} c^{k,l}_{z,w}\, y^w \log^l y$$

and a similar expansion for $b_{w,l}(x)$ as $x \to 0$, with the same coefficients $c^{k,l}_{z,w}$. We denote this fact somewhat loosely by

$$u(x,y) \overset{x,y\to 0}{\sim} \sum_{(z,k)\in E} \sum_{(w,l)\in F} c^{k,l}_{z,w}\, x^z \log^k x\ y^w \log^l y \tag{36}$$

and call this the *corner expansion* of u.

If u is compactly supported (we denote this by $u \in \mathcal{A}^{E,F}_c(\mathbb{R}^2_+)$) then we want to integrate u over the parabola $xy = t$ and consider the asymptotics as $t \to 0$. Integration makes sense only for densities (not for functions). We will write these always as b-densities, that is, we write $u(x,y)\frac{dx}{x}\frac{dy}{y}$, not $u(x,y)dxdy$. Of course one can be rewritten as the other by changing u, but in this way the formulas will be simplest.

Thus, for $u \in \mathcal{A}^{E,F}_c(\mathbb{R}^2_+)$ we define $v : (0,\infty) \to \mathbb{C}$ via the push-forward under the map $f : \mathbb{R}^2_+ \to \mathbb{R}_+,\ (x,y) \mapsto xy$,

$$v(t)\frac{dt}{t} = f_*(u(x,y)\frac{dx}{x}\frac{dy}{y}),$$

or explicitly for $t > 0$

$$v(t) = \int_0^\infty u(x, \frac{t}{x})\frac{dx}{x}. \tag{37}$$

First, we have as a special case of the Push-Forward Theorem of R. Melrose:

$$v \in \mathcal{A}^{E\overline{\cup}F}_c(\mathbb{R}_+) \tag{38}$$

where the **extended union** of index sets E, F is defined as

$$E\overline{\cup}F := E \cup F \cup \{(z, k+l+1) : (z,k) \in E,\ (z,l) \in F\}. \tag{39}$$

That is, v inherits from u all its asymptotic terms as $x \to 0$ or as $y \to 0$, and in addition gets a $t^z \log^{k+l+1} t$ whenever u has $x^z \log^k x$ and $y^z \log^l y$. ('same power, then multiply logs and take one extra')

To get a formula for the coefficients we need the notion of regularized integral, which was introduced before. Recall that if $\inf E > 0$ (that is, $(z,k) \in E \Rightarrow \operatorname{Re} z > 0$, so $v \in L^1(\mathbb{R}_+, \frac{dt}{t})$) then the regularized integral coincides with the standard integral.

Theorem 2.25. *Let $u \in \mathcal{A}_c^{E,F}(\mathbb{R}_+^2)$. Then v, defined in (37), has the $t \to 0$ expansion*

$$
\begin{aligned}
v(t) \sim & \sum_{\substack{z,k,l:\\ (z,k)\in E,(z,l)\in F}} \frac{k!\,l!}{(k+l+1)!} c_{zz}^{kl}\ t^z \log^{k+l+1} t^{-1} \\
& + \sum_{(z,k)\in E} \sum_{m=0}^{k} \left((-1)^m \binom{k}{m} \fint_0^\infty a_{z,k}(y) \frac{\log^m y}{y^z} \frac{dy}{y} \right) t^z \log^{k-m} t \\
& + \dots
\end{aligned}
$$

where the dots mean the same as the previous term, with E, y, a replaced by F, x, b respectively.[10]

Remark 2.26. *This looks complicated but is really rather simple:*

1. *The asymptotics has a contribution from the corner $x = y = 0$ and contributions from each side $x = 0$, $y = 0$.*

   a) *Corner terms: Each 'diagonal' (i.e. the powers of x and y coincide) term $c\,x^z \log^k x\, y^z \log^l y$ in the corner expansion of u contributes $-c\,t^z \log^{k+l+1} t$ (times a constant).*

   b) *Side terms: The contribution of the term $a(y)\, x^z \log^k x$ at the side $x = 0$ can be formally obtained as follows: Replace $x = \frac{t}{y}$ and (reg.) integrate over y:*

   $$a(y)(\frac{t}{y})^z \log^k \frac{t}{y} = t^z\, a(y) y^{-z} (\log t - \log y)^k$$

   Multiplying out, we see that the coefficient of $t^z \log^{k-m} t$ is $(-1)^m \binom{k}{m} a(y) \frac{\log^m y}{y^z}$.

   c) *A simple special case is where u vanishes in a neighborhood of $(0, 0)$: Say $u(x, y) = 0$ for $(x, y) \in [0, c]^2$, then for $t \leq c^2$ we have*

   $$v(t) = \int_c^\infty u(x, \frac{t}{x})\, \frac{dx}{x} + \int_c^\infty u(\frac{t}{y}, y)\, \frac{dy}{y}$$

   (for a proof make a sketch of the parabola $xy = t$). Now the expansion (34) holds uniformly in $x \geq c$, so we may set $y = \frac{t}{x}$ there and integrate term by term to obtain the t-expansion of the first integral, and similarly for the second.

   This yields the formula of the theorem, with vanishing corner terms and regularized integrals being standard integrals. It also explains the formula for the integrand.

   d) *So all that remains to prove the theorem is: explain the corner terms, and explain why the regularization needed for the boundary terms is precisely the one introduced in the previous section.*

2. *Things simplify even more if instead of $\log^k$ we use $\frac{\log^k}{k!}$ everywhere (both in the expansion of u and of v). Then all the factorials go away!*[11]

[10] see Gohar Harutyunyan, An example on asymptotic on manifolds with corners, ARMENIAN JOURNAL OF MATHEMATICS Volume 3, Number 1, 2010, 1-13; here it is assumed that $z, w \in \mathbb{Z}$ always.

[11] This is most probably related to the fact that with $L_k(x) = \frac{\log^k x}{k!}$ we have $x\partial_x L_k = L_{k-1}$. Probably a 'good' proof should use this.

3. An important special case is where u is smooth on $\mathbb{R}^2_+$. If

$$u(x,y) \sim \sum_{z,w=0}^{\infty} c_{zw} x^z y^w$$

is its Taylor expansion at zero, so $c_{zw} = \frac{1}{z!w!}\partial_x^z \partial_y^w u(0,0)$, and

$$u(x,y) \overset{x\to 0}{\sim} \sum_{z=0}^{\infty} a_z(y) x^z$$

$$u(x,y) \overset{y\to 0}{\sim} \sum_{w=0}^{\infty} b_w(x) y^w$$

are the side expansions, so $a_z(y) = \frac{1}{z!}\partial_x^z u(0,y)$, $b_w(x) = \frac{1}{w!}\partial_y^w u(x,0)$, then

$$v(t) \sim \sum_{z=0}^{\infty} t^z(-c_{zz}\log t + A_z + B_z)$$

$$A_z = ⨍_0^\infty \frac{a_z(y)}{y^z}\,\frac{dy}{y},\ \ B_z = ⨍_0^\infty \frac{b_z(x)}{x^z}\,\frac{dx}{x}$$