Lecture Notes: Singular Analysis

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Preface

These are lecture notes for the graduate course 'Singular Analysis' which I give at University of Oldenburg – and online – in the winter term 2020/2021. The plan is to introduce in some detail the basic concepts of R. Melrose's geometric and unifying approach to microlocal analysis on singular and non-compact spaces – manifolds with corners, blow-ups, resolutions, polyhomogeneous functions, push-forward theorem etc. – and to illustrate the power of these concepts for the analysis of PDE in some settings, probably including infinite cylindrical ends/conical singularities (b-calculus) and others, to be determined. The lecture notes will trickle out by and by as we go on. My intention is also that these lecture notes should later become part of a book on the subject (coauthored with Pierre Albin). I am grateful if you send me your suggestions/corrections if you have any, at daniel.grieser@uni-oldenburg.de.

Chapter 1

Introduction

What is singular analysis? Analysis is full of singularities of various kinds. Some examples of this are given below. The goal in this course is to introduce a systematic theory for analyzing PDEs having some sort of singularity. You may think of this as a singularity (non-smoothness) in the coefficients, or as a degeneracy in the coefficients, or as the PDE acting on functions on a singular space, or on a non-compact space.

In fact, these are all just different perspectives of the same thing. As a simple illustration for this claim we consider the Laplacian on a plane sector given by $r > 0$, $0 < \theta < \alpha$ in polar coordinates (r, θ), where $\alpha \in (0, 2\pi)$ is a given opening angle. The sector is a singular space, having a conical singularity at $r = 0$, and its closure is non-compact, as r may go to infinity. In polar coordinates the Laplacian is

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$$

whose coefficients are singular at $r = 0$. We could rewrite this as

$$\Delta = r^{-2}P\,, \quad P = r^2\partial_r^2 + r\partial_r + \partial_\theta^2$$

and the coefficients of P degenerate at $r = 0$, in the sense that P is elliptic, but not uniformly elliptic as $r \to 0$. Note that $\Delta u = f$ is equivalent to $Pu = r^2 f$, so analyzing Δ is equivalent to analyzing P. Finally, the substitution $r = \frac{1}{s}$ turns Δ into

$$\Delta = (s^2\partial_s)^2 - s^3\partial_s + s^2\partial_\theta^2$$

which is again degenerate at $s = 0$, i.e. $r = \infty$, even after taking out a factor of s^2. This shows that non-compactness and singularity/degeneration are very closely related.

A very similar example is analyzed in some detail in Section 1.2, to introduce some important ideas that will guide us throughout the course.

The point of view taken in this course is that the central object to analyze is an operator with smooth but possibly degenerate coefficients, on a non-singular compact space. In the example above the space is $[0,\infty]_r \times [0,\alpha]_\theta$ – a simple example of a manifold with corners – and the operator is P. This will yield a very systematic theory that unifies many different sorts of problems.

We generally assume that this degeneration has some sort of finite structure. In the example above it could be described by saying that ∂_r always occurs in the combination $r\partial_r$. A general definition of 'finite structure' is R. Melrose's notion of boundary fibration structure. Besides the conical singularity and 'large end of a cone' non-compactness in the example above, these structures include, for example, polyhedral (iterated edge) singularities, hyperbolic funnels and cusps, also iterated versions of these. Conjecturally they also include suitable resolutions of algebraic varieties, equipped with the Fubini-Study metric, but this has been proven only in low dimensions.

Besides PDE with singular/degenerate coefficients the same set of ideas can be used to analyze families of PDE depending on a parameter, and degenerating as the parameter approaches some limit (so-called singular perturbation problems). I have not decided yet whether to include examples of this in the course.

Note that we do not consider problems involving finite orders of smoothness in the coefficients, e.g. elliptic operators with L^∞ or Hölder continuous coefficients. This is an important field of study with many applications especially to non-linear PDE. Assuming smooth (infinitely often differentiable) coefficients allows us to focus on the effects of degeneration. The restriction to smooth objects also allows us to use the tools of microlocal analysis.

Note that we only deal with linear PDE. The reason is that most of the problems treated here are already hard enough for these, and anyway they form the basis for the study of non-linear PDE. The theory has been applied to non-linear problems by several authors, e.g. for geometric flows or in relativity.

The theory presented in this course is essentially due to Richard Melrose, and has been developed further by many others, often in collaboration with him. See Section 1.3 for a few pointers to the literature. I am very grateful to Richard Melrose for enriching mathematics with these beautiful ideas and for introducing me to this world.

1.1 What is a singularity?

Singularities appear in many parts of mathematics, but there is no general definition. As a rough idea, a singularity is a place where a mathematical object behaves differently than at most other places. Put differently, singular is the opposite of regular, and regular means well-behaved in some way, usually expressible by the existence of a simple local model. Let us look at various types of objects that may exhibit singularities.

- **Spaces.**

  As a first idea, for a space regularity means being a smooth (C^∞) manifold. The local model is $\mathbb{R}^n$ (by definition of a manifold). For a space sitting inside a larger (regular) space we may define regularity as being a submanifold, then the local model is a vector subspace in $\mathbb{R}^n$. The space may also carry a metric, then the singularity may reside in the metric. Here are some ways how singular spaces arise:

  - Level sets of maps/solution sets of equations. For example, the set of $(x, y, z) \in \mathbb{R}^3$ satisfying $x^2 + y^2 - z^2 = 0$ is a double cone. A little more interesting is the Whitney umbrella, obtained from $x^2 = y^2 z$. In general, polynomial equations define algebraic varieties, a very rich source of singular spaces of great significance in mathematics.
  - Quotients of non-free group actions. For example, the orbits of the rotation action of S^1 on $\mathbb{R}^2$ are the sets $\{|x| = r\}$ where $r \geq 0$. This is a circle if $r > 0$ and a point if $r = 0$. So the quotient $\mathbb{R}^2/S^1$, i.e. the space of orbits, can be identified with the closed half line $[0, \infty)$. The fact that this is not a manifold at 0 corresponds to the fact that the action behaves differently at $0 \in \mathbb{R}^2$ (it is not free) than everywhere else.
  - Solutions of geometric PDE. For example, the Schwarzschild spacetime is a solution of Einstein's equation of general relativity, and it has a singularity at the origin. Solutions of the Ricci flow equation usually develop singularities after a finite time interval, and this was a major stumbling block in the proof of Poincar's conjecture, finally overcome by Perelman.
  - There are good reasons to consider only *compact* manifolds as regular, so the idea of regularity has a flavor of finite structure. Here is an example why this may be reasonable: Consider the graph of the function $\sin x$ on $\mathbb{R}$. While it is perfectly smooth, its behavior at infinity is, after the change of variables $x \mapsto \frac{1}{x}$, the same as that of $\sin \frac{1}{x}$ near zero, which you would probably think of as singular. So the non-compactness of $\mathbb{R}$ allows that a nice smooth function has 'singular' behavior.

    The example in the beginning of the introduction also shows why it may be reasonable to consider the non-compactness of the sector at $r = \infty$ as a singularity.
  - A more sophisticated example of current interest are moduli spaces. For example, the Riemann moduli space, i.e. the space of conformal structures on a given compact Riemann surface, has natural compactifications which are singular spaces.

  In this book, also manifolds with corners are considered as regular, for reasons that will become clear by and by.

- **Smooth maps.**

  Recall that for a smooth map $f : \Omega \to \Omega'$, where $\Omega \subset \mathbb{R}^n$, $\Omega' \subset \mathbb{R}^k$ are open, a point $p \in \Omega$ is called a regular point if $df_{|p} : \mathbb{R}^n \to \mathbb{R}^k$ is surjective, and a point $q \in \Omega'$ is called a regular value if all $p \in f^{-1}(q)$ are regular points. If q is a regular value then $f^{-1}(q)$ is a submanifold, so this is closely related to the first example above of singular spaces. The opposite is then a singular point or singular value.

  The local model at a regular point is a projection, by the implicit function theorem: If $p \in \Omega$ is regular then one can introduce coordinates on neighborhoods of p and $f(p)$ in terms of which f is projection to the first k coordinates.

- **Smooth vector fields.**

  If $V : \Omega \to \mathbb{R}^n$ is a smooth vector field on an open subset $\Omega \subset \mathbb{R}^n$ then $p \in \Omega$ is called a singular point of V if $V(p) = 0$.

  The local model near a regular point is a constant vector field: coordinates near p can be introduced in terms of which $V = \frac{\partial}{\partial x_1}$. In particular, the flow of V near a regular point is simple to understand. On the other hand, the flow near a singular point can be quite wild. This is one major stumbling block in the resolution of Hilbert's 16th problem, which is still outstanding.

- **Functions.**

  In complex analysis, one classifies isolated singularities as removable, pole or essential. While poles still have simple models (z^n), essential singularities can be wild.

  In distribution theory there is the notion of singular support – the set of points where a distribution is not locally given by a smooth function. (So the local model is smoothness.) The singular support, and the distribution, can be quite tame (as e.g. in the case of conormal distributions), or arbitrarily wild.

  These first two examples are less relevant for this book (conormal distributions will appear, but will be thought of as rather regular objects).

  More along the lines of this book is the following example: Consider $f(x, y) = \frac{x}{x+y}$ on $\mathbb{R}^2_+ \setminus \{0\}$. Then f is 0 on the y-axis and 1 on the x-axis. So it cannot have a continuous extension to $x = y = 0$. However, as we will see, when we blow up this point a continous extension to the boundary exists.

- **Differential operators.** For a linear ordinary differential operator of order m

$$P = a_m(x)\partial_x^m + \ldots a_1(x)\partial_x + a_0(x)$$

with smooth coefficients a_j a point x is called regular if $a_m(x) \neq 0$ and singular otherwise. The Picard-Lindelöf theorem guarantees that the equation $Pu = 0$ has m independent smooth solutions in a neighborhood of a regular point. A simple example of a singular point is the point zero for

$$P = x\partial_x - c\,,\ c \in \mathbb{C}$$

and $Pu = 0$ has the solution $u(x) = x^c$, which is non-smooth at zero if $c \notin \mathbb{N}_0$. This example is fundamental for the b-calculus.

1.2 A PDE example, and an outlook

To get an idea where we are heading, consider the cone

$$M = \{(x, y, z) \in \mathbb{R}^3 : z > \sqrt{x^2 + y^2}\}$$

(see Figure xxx). Its boundary ∂M has a conical singularity at 0. Denote the regular part of the boundary by $\partial_{\mathrm{reg}} M = \partial M \setminus \{0\}$. We consider the PDE problem

$$\Delta u = f \text{ on } M, \quad u = 0 \text{ on } \partial_{\mathrm{reg}} M\,, \tag{1.1}$$

where Δ is the Euclidean Laplacian and f is a given function on M. Here are some typical questions we might ask:

1. What is the behavior of solutions u near the singularity 0?
2. What is the behavior of solutions u at infinity?
3. What are properties of solution operators $f \mapsto u$?
4. Existence, uniqueness of solutions?

These questions can be answered quite explicitly due to the special structure of the problem, which allows us to separate variables. However, some of the insights gained from doing this can be transferred to more general problems where separation is not possible. This will lead us naturally to some of the fundamental concepts used in this book, as we explain below. We will return to this model problem in later chapters to illustrate the progress we make.

Polar coordinates and separation of variables

It is natural to introduce spherical polar coordinates: write $p \in M$ as $p = r\omega$ with $r > 0$ and $\omega \in \Omega \subset S^2$, where $S^2 = \{\omega \in \mathbb{R}^3 : |\omega| = 1\}$ and $\Omega = M \cap S^2$. We write u in polar coordinates, i.e. we set $\tilde{u}(r, \omega) = u(r\omega)$ and similarly for f. In these coordinates Δ turns into

$$\tilde{\Delta} = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta^S \tag{1.2}$$

with Δ^S the Laplace-Beltrami operator on the sphere S^2.[1] Note that

$$\tilde{\Delta} = r^{-2} P\,, \quad P = (r\partial_r)^2 + r\partial_r + \Delta^S\,. \tag{1.3}$$

So we have

$$\Delta u = f \iff \tilde{\Delta}\tilde{u} = \tilde{f} \iff P\tilde{u} = r^2\tilde{f}\,.$$

The form of P in (1.3) cries out for using separation of variables, and this leads us to first consider the eigenvalue problem

$$-\Delta^S\phi = \lambda\phi \text{ in } \Omega, \quad \phi = 0 \text{ in } \partial\Omega\,.$$

It is a standard fact that there is a sequence of solutions (λ_j, ϕ_j) where $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty$, the ϕ_j are smooth and form an orthonormal basis of $L^2(\Omega)$. We expand $\tilde{u}(r,\cdot)$ and $r^2\tilde{f}(r,\cdot)$ in this basis:

$$\tilde{u}(r,\omega) = \sum_j a_j(r)\phi_j(\omega)\,, \quad r^2\tilde{f}(r,\omega) = \sum_j b_j(r)\phi_j(\omega) \tag{1.4}$$

Ignoring questions of convergence for the moment, we obtain that $\tilde{\Delta}\tilde{u} = \tilde{f}$ is equivalent to the set of ODEs

$$P_j a_j = b_j\,, \quad P_j = (r\partial_r)^2 + r\partial_r - \lambda_j \tag{1.5}$$

for all j. The solutions of the homogeneous equation (i.e. $b_j = 0$) are[2] linear combinations of

$$a_j^{\pm}(r) = r^{z_j^{\pm}}\,, \quad z_j^{\pm} = -\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} + \lambda_j}\,. \tag{1.6}$$

Behavior near 0 and near ∞

From (1.6) we can read off the behavior of solutions u of (1.1) as $r \to 0$: If f vanishes near 0 then u is a sum of terms $r^{z_j^{\pm}}\phi_j(\omega)$, when written in polar coordinates. Note two things about this:

- The $z_j^{\pm}$ need not be integers. So the asymptotics of u as $r \to 0$ is similar to, but more general than a Taylor expansion.
- $z_j^- < 0$, so solutions can blow-up as $r \to 0$. If we impose additional conditions on u (e.g. boundedness) then this rules out these terms.

More generally, if $\tilde{f}$ behaves like a sum of terms $r^z\phi_j(\omega)$ as $r \to 0$, then u will, in addition, have terms r^z and $r^z \log r$ (if z equals some $z_j^{\pm}$) times ϕ_j.

Analogous remarks apply for $r \to \infty$.

[1] One could write Δ^S explicitly in terms of spherical coordinates as $\frac{1}{\sin\theta}\partial_\theta \sin\theta\,\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2$, but this introduces artificial singularities at the poles, and in any case we don't need this.

[2] This is an Euler type equation, make the ansatz $a(r) = r^z$. Alternatively, substitute $r = e^s$, then it becomes a constant coefficient ODE.

Solution operators

Let us look for a solution operator in the form

$$(Qf)(p) = \int_M K(p;p')f(p')\,dp'\,. \tag{1.7}$$

That is, $u = Qf$ should be a solution of (1.1) for any f. Of course this can only work if we impose some restrictions on f to ensure that the integral converges. In fact, there will be many such operators, and separation of variables allows us to write them down quite explicitly: In polar coordinates (1.7) reads

$$(\tilde{Q}\tilde{f})(r,\omega) = \int_0^\infty \int_\Omega \tilde{K}(r,\omega;r',\omega')\tilde{f}(r',\omega')d\omega'\,(r')^2dr'$$

where $d\omega'$ is surface measure on the sphere. Now take[3]

$$\tilde{K}(r,\omega;r',\omega') = \sum_j K_j(r,r')\phi_j(\omega)\overline{\phi_j(\omega')} \tag{1.8}$$

where

$$(Q_jb)(r) = \int_0^\infty K_j(r,r')b(r')\,(r')^2dr'$$

is a solution operator for the ODE (1.5) for each j. To find K_j simply solve $P_jK_j = \delta(r-r')$ for each fixed $r' > 0$. Substituting $t = \frac{r}{r'}$ and using the Euler structure of P_j we get a solution

$$K_j(r,r') = \frac{1}{r'}\,k_j(\frac{r}{r'}) \tag{1.9}$$

where k_j solves $P_jk_j = \delta_1$ and can be explicitly determined.

Existence, uniqueness

Again, this is easy to understand using separation of variables (1.4). Since functions u with coefficients $a_j^\pm(r) = r^{z_j^\pm}$ solve the homogeneous equation $\Delta u = 0$ there is a high degree of non-uniqueness. However, since each $r^{z_j^\pm}$ tends to infinity as $r \to 0$ or as $r \to \infty$, uniqueness can be achieved by imposing suitable growth conditions on u at both ends. Similarly, a solution of the equation $\Delta u = f$ exists within a class of such restricted u if corresponding restrictions are imposed on f.

More generally, Δ has finite-dimensional kernel and cokernel if considered as acting between spaces with polynomial growth conditions.

The non-uniqueness can also be seen for the solution operators: for each j one can add $r^{z_j^+}\,\alpha_+(r') + r^{z_j^-}\,\alpha_-(r')$ to K_j for arbirary functions $\alpha_\pm$, at the price of getting growth at $r = 0$ or $r = \infty$.

[3]Ignoring convergence again; the sum does converge, in the sense of distributions, for reasonable choices of the K_j.

How to generalize this?

Separation of variables is a rather rigid method: it is limited to problems with some symmetry (in our case, dilation invariance). However, many of the results obtained above can be extended to more general problems, but this requires more flexible methods. We now give a sketch of such methods for the problem (1.1) with the same cone M and with Δ replaced by an elliptic variable coefficient operator

$$A = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} b_i(x)\partial_{x_i} + c(x)$$

where all coeffients are smooth on $\mathbb{R}^n$ and

$$a_{ij}(x) \to \delta_{ij}\,,\ b_i(x) \to 0\,,\ c(x) \to 0 \quad \text{as } x \to \infty \tag{1.10}$$

sufficiently fast.[4] This still has the salient features of the original problem (a conical singularity at 0, an infinite cone type of non-compactness, and uniform ellipticity up to the boundary) but lacks the dilation invariance. Then we will identify which structures of the problem are essential for these methods to work, and how these lead to the main ingredients of the theory to be developed in this book.

Let us consider the questions above:

- **Behavior of solutions at 0:** Although separation of variables won't work for our generalized problem, it will work to leading order at $r = 0$, and this is enough for finding the complete asymptotic expansion of solutions. This will lead us to the idea of **model operator**.

  When writing A in polar coordinates we get (exercise!)

  $$\tilde{A} = r^{-2}P\,, \quad P = P_0 + rP_1\,, \quad P_0 = (r\partial_r)^2 + r\partial_r + \Delta^{S'}$$

  where $\Delta^{S'}$ is the spherical Laplacian after a linear change of coordinates[5] in $\mathbb{R}^3$ and P_1, like P_0, is built from

  $$r\partial_r\,, \quad \partial_{\omega_i} \tag{1.11}$$

  where ∂_{ω_i} denote angular derivatives, with coefficients smooth in $r \geq 0$. We can now formally construct a solution of $Pu = 0$: Note that for any $z \in \mathbb{C}$ and $v \in C^\infty(S^2)$

  $$\begin{aligned} P_0(r^z v) &= r^z w & w &= w(\omega) = P_0^z v\,,\ P_0^z = z^2 + z + \Delta^{S'} \\ rP_1(r^z v) &= r^{z+1} w' & w' &= w'(r,\omega) \text{ smooth in } r \geq 0. \end{aligned}$$

[4] Ellipticity here means that the matrix $(a_{ij}(x))_{i,j}$ is positive definite for each x, and sufficiently fast you can take to mean that the errors in (1.10) are $O(|x|^{-2})$, though weaker conditions suffice. We will also impose similar conditions on the derivatives. Geometrically, A could be the Laplace-Beltrami operator for a metric with conical singularity at one point and asymptotically conic behavior at ∞, where the base of the cone has a boundary.

[5] $\Delta^{S'} = \Delta^S$ if $a_{ij}(0) = \delta_{ij}$.

Therefore, we can find a formal solution[6]

$$u = r^z v_0 + r^{z+1} v_1 + \dots \tag{1.12}$$

(with $v_0, v_1, \dots \in C^\infty(S^2)$) by solving $P_0^z v_0 = 0$, $P_0^{z+1} v_1 = -w_0'(0,\cdot)$ etc. The first of these equation requires that v_0 be an eigenfunction of $\Delta^{S'}$ and that z solve $z^2 + z - \lambda = 0$ where λ is the eigenvalue.[7]

Note that P_0 encodes the leading behavior of A at $r = 0$, so it is called the **model operator** of A at $r = 0$.[8] Important: all the equations that need to be solved here only involve inverting P_0. This can be done since P_0 is dilation invariant in r, so it reduces to inverting the ordinary differential operators P_0^{z+k}.

Again, the inhomogeneous problem can be formally solved in a similar way.

- **Behavior of solutions at ∞:** We substitute $s = \frac{1}{r}$, then $r \to \infty$ corresponds to $s \to 0$. Then $\partial_r = -s^2 \partial_s$, so for $A = \Delta$ we get from (1.2)

$$\tilde{\Delta} = (s^2 \partial_s)^2 - 2s^3 \partial_s + s^2 \Delta^S \tag{1.13}$$

If the coefficients of A are 'smooth at infinity' (i.e. smooth in s, ω at $s = 0$)[9] and if (1.10) holds then we can write $\tilde{A} = P_0' + sP_1'$ where now $P_0' = \tilde{\Delta}$ and P_1', like P_0', is built from

$$s^2 \partial_s \,, \quad s\partial_{\omega_i} \tag{1.14}$$

with coefficients smooth in $s \geq 0$. So $\tilde{\Delta}$ is the **model operator at ∞**. Observe two things about this:

1. We can factor out s^2; then $\tilde{A} = s^2 \tilde{B}$ where $\tilde{B}$ has the same structure as $s \to 0$ as $\tilde{A}$ had for $r \to 0$. So the analysis above applies, and we get formal solutions behaving as in (1.12) with r replaced by r^{-1}.
2. If the last condition in (1.10) is replaced by $c(x) \to C$ for $x \to \infty$, where C may be non-zero, then we cannot factor out s^2 from $\tilde{A}$. The model operator will now be $\tilde{\Delta} + C$, and as we will see in the chapter on the scattering calculus the solution behavior for this operator is completely different from that in the case $C = 0$ (and depends heavily on the sign of C as well).

- **Solution operators:** Of course integral kernels K of solution operators Q as in (1.7) cannot be calculated explicitly. However, the classical pseudodifferential operator (ΨDO) calculus[10] yields **parametrices** (approximate

[6] Meaning that for any $N \in \mathbb{N}$, if u_N is the sum of the first N terms, $\tilde{A}\tilde{u} = O(r^{z+N})$ as $r \to 0$.

[7] The next equation, $P_0^{z+1} v_1 = -w'(0,\cdot)$, is solvable in general only if $(z+1)^2 + (z+1)$ is not an eigenvalue of $\Delta^{S'}$. However, even if it is then we can get formal solutions in the more general form $u = r^z v_0 + r^{z+1}(\log r) v_1 + \dots$. This will be discussed in detail in the chapter on the b-calculus.

[8] In this specific context it will be called the indicial operator of $\tilde{A}$.

[9] This corresponds to similar conditions as (1.10) for all derivatives of the coefficients.

[10] This will be reviewed in Chapter XXX.

solution operators) with a precise description of the behavior of $K(p,p')$ at the diagonal, i.e. as $p \to p'$, uniformly for p, p' in any compact subset of M, i.e. staying away from 0, ∞ and $\partial_0 M$: at the diagonal K has a singularity whose leading term is the Newton potential $\frac{1}{|p-p'|}$, and with a full asymptotic expansion as $p \to p'$ (most efficiently expressed in terms of the Fourier transform). Outside of the diagonal K is smooth.

The question then is how to refine the construction of the parametrix so that it correctly models the behavior uniformly as p, p' approach 0 (or ∞ or $\partial_0 M$).[11] A natural guess is that the leading order behavior should be that of the model operator, i.e. as in (1.8), (1.9). Thus, the pseudodifferential calculus needs to be extended to encode this kind of behavior, and similarly at ∞ or $\partial_0 M$.

- **Existence and uniqueness:** These are global questions and usually cannot be answered directly.[12] For example, in the case of Δ we used the fact that $r^{z_j^\pm}\phi_j(\omega)$ are global solutions, and uniqueness depended on the behavior of $r^{z_j^\pm}$ at both 0 and ∞. For general A we don't know global solutions like this.

We sketched how to obtain formal solutions at 0 and ∞ and parametrices (approximate solution operators). This leaves open the question whether all solutions have to behave like this, and whether there are actual solution operators having a similar behavior as the parametrices. See below for more on this.

Essential structures and outlook

We now identify the essential structures that allowed us to carry out these steps, and point out how they lead to some of the fundamental concepts used in this book.

(a) Separation of variables or its generalization, the iterative solution using the model operator, only work because we introduced polar coordinates. Using these we transferred the problem from M to $\tilde{M} = (0,\infty)_r \times \Omega$. Since we are interested in the behavior at $r = 0$, $r = \infty$ and $\partial_0\tilde{M} = (0,\infty)\times\partial\Omega$, it is natural to *compactify* and consider $X = [0,\infty]\times\bar{\Omega}$ as underlying space of the problem.[13]

The main point is that X has a **local product structure** near $r = 0$ and $r = \infty$.[14]

[11] Technically, this means that the (integral kernels of the) error terms R, R' in the 'approximate inverse equations' $AQ = I + R$, $QA = I + R'$ should be small uniformly in p, p'.

[12] In some special circumstances special arguments like positivity/maximum principle can be used to prove uniqueness.

[13] The notation $[0,\infty]$ requires explanation. In this context smoothness of functions at ∞ means smoothness in terms of $s = \frac{1}{r}$ up to $s = 0$. In this way $[0,\infty]$ becomes a smooth manifold with boundary.

[14] In fact, it has a global product structure, but this is *not* essential.

Spaces that locally near the boundary have product structures are called **manifolds with corners**. So the first step in our solution was to replace the singular space M by the manifold with corners X.

The transition from M to X has two non-trivial aspects: at $r = 0$ it is given by **blowing up** the point 0; at $r = \infty$ it is given by the socalled **radial compactification**. These are closely related since both correspond to adding an endpoint ($r = 0$ or $r = \infty$) for each direction ω. Blow-up is one of the fundamental operations of singular analysis. [15]

(b) By replacing M by X we removed the singularity (and non-compactness) from the underlying space. But the singularity lives on in the structure of the operator $\tilde{\Delta}$, which is encoded in the expressions (1.11) near $r = 0$, resp. (1.14) near $r = \infty$.[16]

These expressions, and their linear combinations with smooth coefficients, are vector fields, and in general we will encode singular (or degeneration) structures of operators by such **spaces of vector fields**, usually denoted $\mathcal{V}$.[17]

The pair $(X, \mathcal{V})$ is really the central object of interest. It is called a **boundary fibration structure**.

(c) We have seen the occurence of functions having expansions as $r \to 0$ involving terms like $r^z a(\omega)$ (and possibly logarithms of r). This will lead to the notion of **polyhomogeneous function**.

(d) We observed the central role of **model operators**. So we need tools to invert these. Above we used separation of variables, and for the conical singularity we got power type (r^z) behavior. We will redo this in a slightly different way that fits better into the general framework, using the **Mellin transform**. In addition to the model operators at $r = 0$ and $r = \infty$, the operator A_p obtained by freezing the coefficients of A at a point p and just taking the leading part should also be considered as a model operator. Since it has constant coefficients, it can be inverted using the Fourier transform. This is the starting idea for the **ΨDO calculus** since putting together all these inverses we obtain an (interior) ΨDO parametrix.

(e) Since the integral kernel K of a parametrix depends on two spatial variables p, p', it will be defined on the (interior of the) **double space** $X^2 := X \times X$. We saw that it is essential to understand the expected behavior of $K(p, p')$ when (p, p') approaches the boundary of X^2. Equation

[15] To rename 'introducing polar coordinates' to 'blow-up' may seem superfluous, but is actually very useful since it diverts attention from coordinates to the essential geometric structures. Also, sometimes we have to iterate blow-ups, and expressing that in coordinates would easily drive you insane.

[16] The behavior near the regular boundary $\partial_0 M$ could be encoded similarly, we will get back to this in the chapter on the 0-calculus.

[17] Since they are closed under the Lie bracket of vector fields – and this is quite fundamental –, they are usually called **Lie algebras of vector fields**. Depending on the type of degeneration $\mathcal{V}$ is adorned with extra letters, like $\mathcal{V}_{\mathrm{b}}$, $\mathcal{V}_{\mathrm{sc}}$ etc.

(1.9) shows that we cannot expect it to have 'product type' behavior near $r = r' = 0$, and we will see that this means that we need to **blow up** the corner $r = r' = 0$ of X^2. After a similar blow-up at $r = r' = \infty$ we obtain the **resolved double space**, which is the natural carrier of the integral kernels of the calculus.

Then the power of the ΨDO calculus kicks in: once we have constructed a sufficiently good parametrix, we can use it to show that all solutions have expansions like the formal solutions, that the operator A is Fredholm on suitable function spaces, i.e. has finite dimensional kernel and cokernel, and that its Fredholm inverse (a solution operator) has the same asymptotic behavior as the parametrix.

Summarizing, our analysis of a singular (or non-compact) problem will proceed as follows (for elliptic problems):[18]

1. Preparation: resolve the singularities and compactify to transfer the problem to a regular space X (a manifold with corners) so that both X and the operator have a local product structure; identify the underlying space $\mathcal{V}$ of vector fields encoding the singularity.

2. Identify the relevant model problems, and solve them.

3. Build a pseudodifferential calculus adapted to $(X, \mathcal{V})$. This involves several steps, starting with finding a 'resolution' of the double space X^2.

   Then use the ΨDO calculus to derive solvability, regularity, asymtpotics, Fredholmness etc. results.

Of course sometimes you may not be so ambitious to actually aim to understand the Schwartz kernels of solution operators in detail. Maybe you just want to understand the structure of formal expansions of solutions. In this case you can leave out the third step.

Remark 1.2.1. *The same discussion applies when M in problem* (1.1) *is replaced by an open conical subset in any $\mathbb{R}^n$ (and also for more general 'cone type' operators). In two dimensions, if M is a sector of opening angle $\alpha < \pi$ then it may appear that the sector is already resolved since it is a manifold with corners. However, this is not so: it is the joint product structure of space and operator which is needed, and the operator Δ does not have product structure (except if $\alpha = \pi/2$). In particular our use of the word 'corner' differs somewhat from other uses in the literature, e.g. in M. Dauge's 'corner domains'.*

1.3 A few words on the literature

These lecture notes give an introduction to the manifolds with corners framework for singular analysis introduced by R. Melrose. There are lots of papers

[18] A similar procedure applies to the heat equation.

by now where this has been used in various contexts of PDE and global analysis, and where the theory has been extended to more and more settings. I will not even try to give a full account here but will only mention a few: Some of the early papers from the 1980s are [Mel81], [MM83] and [MM87]. Melrose's ICM lecture [Mel91] gives an overview of where the theory is supposed to go. Many details are worked out in [Mel93] (in the case of the b-calculus), and in great generality in the online book [Mel96]. [Mel08] is very readable. [Gri01] is an attempt at an easily accessible introduction to some of the fundamental ideas of the subject (and provides some more links to the literature of the last millenium). A core element in most of these works is the construction of a pseudo-differential calculus (or a related 'heat calculus') adapted to the singular structure. [Gri17] is a survey emphasizing applications that do not involve pseudo-differential operators. One of the many exciting recent developments is the extension of the theory to so-called iterated spaces (see e.g. [ALMP18]).

There are many other approaches to singular analysis problems, usually dealing with more restricted classes of singularities. One of the earliest papers is by V. Kondratyev [Kon63], and a systematic pseudo-differential theory for cones and iterated cones was developed by B.-W. Schulze and his collaborators, see e.g. [Sch91] (see also [LS00] for a comparison with the b-calculus, and for more references).

Chapter 2

Basic notions: Geometry

In this chapter we introduce the basic geometric objects which we need: manifolds with corners, b-maps, vector fields, blow-ups.

You might ask yourself: 'Why manifolds, why geometry? I only care about PDEs on domains in Euclidean space!' The short answer is that even for those we will naturally be lead to work with manifolds, once the domain has singularities or is non-compact. But don't worry: these manifolds will be very concrete objects, and most of the time we will be able to visualize them easily.

Thinking of a function as being defined on a manifold (rather than on $\mathbb{R}^n$) means thinking of it not as an expression in coordinates but as a geometric object: it associates a value to each point, rather than to each n-tuple of coordinates. Thinking like this, we are then free to choose a suitable coordinate system for any given purpose. Often no specific coordinates are distinguished a priori (and choosing some would distract us from more important things); we saw this in the case of the sphere in the example in Section 1.2.

Another concept, central to singular analysis, where this way of thinking is extremely useful, is blow-up: while you might say that blow-up is nothing but introducing polar coordinates, it is better to think of it as a geometric construction on manifolds (with corners) since this liberates us from polar coordinates – these are quite cumbersome for calculations, compared to projective coordinates, say.

As explained in the introduction, manifolds with corners are the 'regular' spaces on which all our objects (functions, differential operators etc.) live. They appear because they have a local product structure everywhere, and compared to manifolds (without corners or boundary) some of the coordinates may be restricted to be non-negative – like the radial variable in polar coordinates. We

emphasize again that manifolds with corners are *not* the singular spaces on which we consider PDEs, see Remark 1.2.1.

Manifolds with corners appear at many places in the theory: not just as carriers of functions, but also, for example, as carriers of Schwartz kernels of operators. The fact that the class of manifolds with corners is closed under product and blow-up makes this a very round and useful theory.

2.1 Manifolds with corners and smooth maps

To get a first impression of what a manifold with corners is, the reader should look at Figure xxx. Manifolds are spaces that are locally like $\mathbb{R}^n$. Manifolds with corners are locally like the model corners

$$\mathbb{R}^n_k := \mathbb{R}^k_+ \times \mathbb{R}^{n-k} \quad \mathbb{R}_+ = [0,\infty),\ 0 \leq k \leq n.$$

In order to make this precise we need to clarify what smoothness means for a function on such a space. The issue is that the standard definition of smoothness requires the domain of the function to be an open subset of $\mathbb{R}^n$, but $\mathbb{R}^n_k$ is not open if $k > 0$.

First, consider the notion of open subset. By definition, a subset $\Omega \subset \mathbb{R}^n_k$ is **relatively open**, or **open in** $\mathbb{R}^n_k$, if for each $p \in \Omega$ there is $\varepsilon > 0$ so that $\{q \in \mathbb{R}^n_k : |p-q| < \varepsilon\} \subset \Omega$. Of course Ω then need not be open as a subset of $\mathbb{R}^n$. But it is a standard fact that Ω is open in $\mathbb{R}^n_k$ if and only if there is an open subset $\Omega' \subset \mathbb{R}^n$ such that $\Omega = \Omega' \cap \mathbb{R}^n_k$. Such an Ω' is called an **extension** of Ω. See Figure xxx.

We use the notation

$$(\mathbb{R}^n_k)^{\mathrm{o}} := (0,\infty)^k \times \mathbb{R}^{n-k}, \quad \Omega^{\mathrm{o}} := \Omega \cap (\mathbb{R}^n_k)^o$$

for any subset $\Omega \subset \mathbb{R}^n_k$. If Ω is open in $\mathbb{R}^n_k$ then Ω^{o} is the interior of Ω considered as a subset of $\mathbb{R}^n$, and it is dense in Ω.

We define smoothness of functions defined on relatively open subsets of $\mathbb{R}^n_k$.

Definition 2.1.1. *Let $\Omega \subset \mathbb{R}^n_k$ be relatively open and $u : \Omega \to \mathbb{C}$. We say that u is **smooth** if*

(a) $u^{\mathrm{o}} := u_{|\Omega^{\mathrm{o}}}$ is smooth, i.e. in $C^\infty(\Omega^{\mathrm{o}})$.

(b) $D^\alpha u^{\mathrm{o}}$ extends continuously to Ω for all multiindices α.

Clearly, if (b) holds then u is uniquely determined by u^{o}. So we usually do not distinguish between u and u^{o} in notation.

There are two equivalent characterizations of smoothness:

1. A function $u^{\mathrm{o}} \in C^\infty(\Omega^{\mathrm{o}})$ on Ω^{o} extends to a smooth function on Ω if and only if $D^\alpha u^{\mathrm{o}}$ is bounded on $K \cap \Omega^{\mathrm{o}}$ for all compact subsets $K \subset \Omega$.[1]

[1] E.g., for $\Omega = \mathbb{R}_+$, to obtain $u(0)$ simply integrate u' from 1 to 0 and add $u(1)$.

2. (Extension theorem)[2] Let Ω' be an extension of Ω. Then u is smooth if and only if u has a smooth extension to Ω', i.e. if there exists $u' \in C^\infty(\Omega')$ so that $u'_{|\Omega} = u$. (See Exercise **??**.)

We say that a map $F : \Omega \to \mathbb{R}^N$ is smooth if $F = (F_1, \ldots, F_N)$ with each $F_j : \Omega \to \mathbb{R}$ smooth. A map

$$F : \Omega_1 \to \Omega_2, \quad \Omega_1 \subset \mathbb{R}^n_{k_1},\ \Omega_2 \subset \mathbb{R}^n_{k_2} \text{ relatively open}, \tag{2.1}$$

is called a **diffeomorphism** if it is bijective and F and F^{-1} are smooth. For a smooth map F, the differential dF_p, a linear map $\mathbb{R}^n \to \mathbb{R}^n$, is defined for any $p \in \Omega_1$, and it is an isomorphism if F is a diffeomorphism, see Exercise **??**. Note that k_1, k_2 can be different here as in Figure xxx. However, there is an important invariant:

Definition 2.1.2. *If $p \in \mathbb{R}^n_k$ then the **codimension** of $p = (x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ is defined as*

$$\operatorname{codim} p := \#\{i : x_i = 0\}.$$

See Figure xxx for examples. Here we use the **general convention for coordinates**: write x_i for the non-negative coordinates and y_j for the unrestricted coordinates of $\mathbb{R}^n_k$. We have:

Lemma 2.1.3. *Diffeomorphisms preserve codimension: For the map F in* (2.1) *and any $p \in \mathbb{R}^n_{k_1}$ we have*

$$\operatorname{codim} p = \operatorname{codim} F(p)\,.$$

So for example the corner $(0,0) \in \mathbb{R}^2_+$ has to be mapped to itself by a diffeomorphism.

Proof. Define the **inward pointing tangent space** at a point $p \in \Omega_1$ as

$$T^+_p\Omega_1 := \{v \in \mathbb{R}^n : \exists \varepsilon > 0\, \exists \gamma : [0,\varepsilon) \to \Omega_1 \text{ smooth}, \gamma(0) = p,\ \dot\gamma(0) = v\}\,.$$

This is itself a model subspace of $\mathbb{R}^n$, and $\operatorname{codim} p$ equals the codimension of the largest vector subspace of $\mathbb{R}^n$ contained in $T^+_p\Omega_1$. Since F maps Ω_1 to Ω_2, its differential dF_p maps $T^+_p\Omega_1 \to T^+_{F(p)}\Omega_2$. Now the claim follows from the fact that $dF_p : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. □

We can now – almost – define manifolds with corners:

Definition 2.1.4. *(a) A **weak manifold with corners**[3] of dimension $n \in \mathbb{N}_0$ is a topological space X which is Hausdorff and paracompact, together with an open cover $(U_i)_{i\in I}$ and for each $i \in I$ a homeomorphism*

$$\phi_i : \tilde U_i \to U_i\,, \quad \tilde U_i \subset \mathbb{R}^n_k \text{ relatively open, for some } k, \tag{2.2}$$

so that all the coordinate change maps $\phi_j^{-1} \circ \phi_i$ are diffeomorphisms.

[2] This is a special case of Whitney's extension theorem [Whi34]. Seeley [See64] proved, for the half space, that the extension operator can be chosen linear and continuous; see also [Mit61]. See [Mel96, 1.4] for the case with corners.

[3] R. Melrose uses the term t-manifold, where t is for 'tied' – or for teardrop, the simplest example of a weak mwc which is not a mwc.

*(b) A **manifold with corners** (mwc) is a weak manifold with corners all of whose boundary hypersurfaces are p-submanifolds.*

Part (a) mimics the definition of a manifold, with $\mathbb{R}^n$ replaced by $\mathbb{R}^n_k$ (for various k). The terms in (b) are explained below, see Definitions 2.1.6 and 2.2.1. Figure xxx shows a mwc and a weak mwc which is not a mwc, so you may try to come up with the definitions yourself. See also Figure xxx for more examples and non-examples, and Exercise **??**.

Remark 2.1.5. *Some authors define manifolds with corners differently. Some use the term mwc for weak mwc, which may seem more natural. However, we impose the submanifold hypothesis since it turns out to be extremely useful and not too restrictive. A slightly different definition of mwc is used by D. Joyce (TODO reference).*

As usual, we call ϕ_i, or U_i, a **local chart** and ϕ_i^{-1} a **local coordinate system**. The collection of all ϕ_i is called an atlas. Smoothness of functions on X and of maps between weak manifolds with corners is defined as usual using the charts, i.e. as smoothness in coordinates. Any other diffeomorphism as in (2.2) is also called a local chart.[4]

We often denote coordinate systems as (x, y) where $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_{n-k})$, and if a coordinate system is fixed then we often identify a point p with its coordinates $(x(p), y(p))$. We call a coordinate system **centered** at $p \in X$ if p has coordinates $0 \in \mathbb{R}^n_k$. Clearly, for any $p \in X$ there is a coordinates system centered at p.

By Lemma 2.1.3, the following definition makes sense.

Definition 2.1.6. *Let X be a weak manifold with corners.*

*(a) The **codimension** of $p \in X$ is defined as the codimension of $\phi_i^{-1}(p)$, for any local chart containing p.*

*(b) A **face** of codimension k of X is the closure of a connected component of $\{p \in X : \operatorname{codim} p = k\}$.*

*(c) A **boundary hypersurface** of X is a face of codimension one.*

See Figure xxx. We denote

$$M_k(X) = \text{ the set of faces of } X \text{ having codimension } k$$

$$M(X) = \bigsqcup_{k \geq 0} M_k(X)$$

$$\partial X = \bigcup_{H \in M_1(X)} H\,, \quad X^{\circ} = X \setminus \partial X$$

Note that $M_0(X)$ is the set of connected components of X. Example 2.3.6 shows why it is a useful convention to count them among the faces of X.

[4] All these charts then define a maximal atlas, and a weak mwc is, strictly speaking, defined by a maximal atlas.

2.2 Submanifolds

Recall that a subset Y of a manifold X is called a submanifold if locally it is a coordinate subspace in suitable coordinates. This has an obvious generalization to manifolds with corners.

Definition 2.2.1. *Let X be a weak manifold with corners. A connected subset $Y \subset X$ is called a **p-submanifold** if for each $p \in Y$ there are coordinates centered at p so that Y is a coordinate subspace in these coordinates.*

*If $Y \subset \partial X$ then Y is a **boundary p-submanifold**, otherwise it is an **interior p-submanifold**.*

Explicitly, the condition is that coordinates $x_1, \dots, x_k, y_1, \dots, y_{n-k}$ centered at p can be chosen on a neighborhood U of p so that

$$Y \cap U = \{(x, y) : x_i = 0 \, \forall i \in I \, , \; y_j = 0 \, \forall j \in J\} \tag{2.3}$$

for some subsets $I \subset \{1, \dots, k\}$, $J \subset \{1, \dots, n-k\}$. For an interior submanifold I is always empty.

Remarks 2.2.2.

- *p is for product: the condition may be restated as saying that locally Y is the zero section in a product neighborhood: after shrinking U if necessary we have $U = U_{I,J} \times U_{I^c,J^c}$ where $U_{I,J}$ denotes the set on the right in* (2.3)*, and $Y \cap U = U_{I,J} \times \{0\}$.*
- *We assume that Y is connected for simplicity. It ensures that either Y is 'everywhere interior' or nowhere.*
- *There are more general notions of submanifolds of a manifold with corners, see [Mel96].*
- *It is easy to see that p-submanifolds are weak mwc themselves.*
- *There is a characterization as local level sets of p-maps, see Exercise ...*

If $p \in X$ has codimension 1 then the unique boundary hypersurface containing p is, near p, a p-submanifold; but the teardrop example shows that it need not globally be a p-submanifold. This issue is closely related to another important concept:

Definition 2.2.3. *Let X be a weak manifold with corners and H a boundary hypersurface. A **boundary defining function** for H is a smooth function $\rho : X \to \mathbb{R}_+$ satisfying*

$$\rho^{-1}(0) = H \, , \quad d\rho_p \neq 0 \text{ for all } p \in H \, .$$

As a simple example, take $X = \mathbb{R}_+$ and $H = \{0\}$. Then the conditions are equivalent to $\rho(0) = 0$, $\rho(x) > 0$ for $x > 0$, and $\rho'(0) > 0$. So $\rho(x) = x$ is a boundary defining function but $\rho(x) = x^2$ is not.

Note that if the first condition is satisfied then $\rho \equiv 0$ on H, so the second condition means that the directional derivative $d\rho_p(v)$ be strictly positive for every strictly inward pointing vector $v \in T_pX$.

Lemma 2.2.4. *Let X be a weak manifold with corners and $H \in M_1(X)$.*

(a) A boundary defining function (bdf) for H can be taken as local coordinate x_1 near any point of H.

(b) If ρ is a bdf for H and $a \in C^\infty(X)$ is strictly positive then $a\rho$ is a bdf for H, and every bdf for H is of this form.

Proof. (a) Let ρ be a bdf for H and $p \in H$. Choose any local coordinate system $x_1, \dots, x_k, y_1, \dots, y_{n-k}$ centered at p. Then H is locally given by the vanishing of one of the x_i, say x_1. Write ρ as a function of x, y, then $d\rho_p \neq 0$ is equivalent to $\frac{\partial \rho}{\partial x_1}(p) \neq 0$. So by the inverse function theorem the set $\rho, x_2, \dots, x_k, y_1, \dots, y_{n-k}$ is a coordinate system near p again.

(b) Clearly, $\tilde{\rho} := a\rho \geq 0$ and $\tilde{\rho}(p) = 0 \iff \rho(p) = 0 \iff p \in H$; also, $d\tilde{\rho} = (da)\rho + a(d\rho) = a(d\rho) \neq 0$ at H, so $\tilde{\rho}$ is a bdf. Conversely, if $\tilde{\rho}$ is any bdf for H then define $a = \frac{\tilde{\rho}}{\rho}$ in $X \setminus H$. The function a is smooth and positive, and we need to check that it extends to a smooth and positive function on all of X. This is a local statement near H, so it suffices to check it in local coordinates (x, y), where we may assume $\rho = x_1$ by (a). Writing $\tilde{\rho}$ as a function of x, y we get that $\tilde{\rho}(x, y) = 0$ if $x_1 = 0$, and Taylor's theorem implies that $\tilde{\rho} = b \cdot x_1 = b\rho$ with b smooth in $x_1 \geq 0$. Then $b = a$ in $x_1 > 0$, so b is a smooth extension of a, and $0 \neq d\tilde{\rho} = bd\rho$ implies $b \neq 0$, so $b > 0$ at H also. □

Lemma 2.2.5. *Let X be a weak manifold with corners and $H \in M_1(X)$. The following are equivalent:*

(i) H is a p-submanifold.

(ii) H has a boundary defining function.

Proof. (ii)⇒(i) follows from Lemma 2.2.4(a) since taking the bdf as x_1 we get that H is locally given by $x_1 = 0$.

To prove (i)⇒(ii) we use a partition of unity: cover H by coordinate neighborhoods $U_i \subset X$, $i \in I$, and let ρ_i be the coordinate vanishing on $H \cap U_i$. Choose an open subset U_0 of X disjoint from H and containing $X \setminus \bigcup_{i \in I} U_i$, and let $\rho_0 \equiv 1$ on U_0. Choose a partition of unity (ϕ_i) subordinate to the cover of X by the U_i, $i \in I \cup \{0\}$, and set $\rho = \sum_{i \in I \cup \{0\}} \phi_i \rho_i$. Then ρ is a bdf for H. □

From Definition 2.1.4 we get:

Proposition 2.2.6. *Let X be a weak manifold with corners. Then X is a manifold with corners if and only if every $H \in M_1(X)$ has a boundary defining function.*

If X is a mwc then every face of X is a mwc.[5] An intersection of k boundary hypersurfaces of X is either empty or a disjoint union of faces of codimension k.

2.3 b-maps

A b-map is a smooth map between manifolds with corners that respects the boundary in a certain sense. To give an idea in which sense we first consider the local model:

Definition 2.3.1. *Let $\Omega \subset \mathbb{R}^n_k$, $\Omega' \subset \mathbb{R}^{n'}_{k'}$ be open and $f : \Omega \to \Omega'$ smooth. Write $f(x,y) = (f_1(x,y), \dots, f_{k'}(x,y), \dots)$. Then f is called a **b-map** if for each $j = 1, \dots, k'$ either*

$$\begin{aligned} &(i) \quad f_j \equiv 0 \quad \text{or} \\ &(ii) \quad f_j(x,y) = a_j(x,y) \prod_{i=1}^{k} x_i^{e_{ij}} \end{aligned} \tag{2.4}$$

with $e_{ij} \in \mathbb{N}_0$ and $a_j > 0$ smooth.

Note that this is only a condition on the first k' components of f, i.e. those corresponding to the x'_j-variables in $\mathbb{R}^{n'}_{k'} = \mathbb{R}^{k'}_+ \times \mathbb{R}^{n'-k'}$, which define the boundaries.

Example 2.3.2. *Consider the maps $\mathbb{R}^2_+ \to \mathbb{R}_+$ given by*

$$f(x_1,x_2) = x_1 x_2\,, \quad g(x_1,x_2) = x_1 + x_2\,.$$

f is a b-map but g is not.

In which way is f 'better' than g? One answer is this: We will be interested in functions like $\log t$ or t^z on $\mathbb{R}_+$, and we will need to pull them back under smooth maps, i.e. consider $\log f$, $\log g$ or f^z, g^z. Now

$$\log f(x_1,x_2) = \log x_1 + \log x_2\,, \quad \log g(x_1,x_2) = \log(x_1+x_2)$$

and $f^z(x_1,x_2) = x_1^z x_2^z$. So $\log f$ and f^z are sums or products of functions of x_1 and of x_2 separately, but $\log g$ is not even a sum of such products, and neither is g^z if $z \notin \mathbb{N}_0$. The generalization of this property of f is the pull-back theorem for polyhomogeneous functions, see XXX.

To globalize the condition on b-maps note that the coordinates x_i are boundary defining functions for $\mathbb{R}^n_k$, and if x'_j denote the boundary defining coordinates for $\mathbb{R}^{n'}_{k'}$ then $f_j(x,y) = x'_j(f(x,y)) = (f^* x'_j)(x,y)$, so $f_j = f^* x'_j$. So a natural global version of Definition 2.3.1 is this:

[5] Proof by induction on the codimension. Use that if F is a face of X then the boundary hypersurfaces of F are components of intersections $F \cap H$ where $H \in M_1(X)$. If ρ is a boundary defining function for $H \subset X$ then its restriction to F is a boundary defining for each such component.

Definition 2.3.3. *Let X, Y be manifolds with corners and $f : X \to Y$ a smooth map. Choose boundary defining functions ρ_G for all $G \in M_1(X)$ and ρ'_H for all $H \in M_1(Y)$.*

Then f is called a ***b-map*** *if for each $H \in M_1(Y)$ either*

$$\begin{aligned} &(i) \quad f^*\rho'_H \equiv 0 \quad \text{or} \\ &(ii) \quad f^*\rho'_H = a_H \prod_{G \in M_1(X)} \rho_G^{e(G,H)} \\ &\qquad \text{with } e(G,H) \in \mathbb{N}_0 \text{ and } a_H > 0 \text{ smooth.} \end{aligned} \tag{2.5}$$

If case (ii) occurs for all $H \in M_1(Y)$ then f is called an ***interior*** *b-map, otherwise it is a* ***boundary*** *b-map.*

By Lemma 2.2.4 the condition is independent of the choice of boundary defining functions, and it is in fact equivalent to the local condition holding everywhere, see Exercise **??**. In concrete situations we usually check the local condition in Definition 2.3.1.

Simple examples of b-maps are projections $X \times Y \to X$ where X, Y are manifolds with corners, and embeddings of p-submanifolds. More interesting examples are blow-down maps, to be introduced in Section **??**.

The composition of b-maps is a b-map again.

Boundary geometry of b-maps

First consider the meaning of (i) in Definition 2.3.3. If $p \in X$ then $(f^*\rho'_H)(p) = \rho'_H(f(p))$, and this is zero iff $f(p) \in H$. This implies:

Lemma 2.3.4 (Boundary b-maps)**.** *Let $f : X \to Y$ be a b-map. Let $\mathcal{H}_0 = \{H \in M_1(Y) : f^*\rho'_H \equiv 0\}$. Then*

$$f(X) \subset Y' := \bigcap_{H \in \mathcal{H}_0} H$$

and the map $f : X \to Y'$ is an interior b-map.

So we only need to study interior b-maps. Note that if $f : X \to Y$ is an interior b-map then $e(G, H)$ is defined for all $G \in M_1(X)$, $H \in M_1(Y)$. Then

$$E = \{(G, H) : e(G, H) > 0\} \subset M_1(X) \times M_1(Y)$$

defines a relation between $M_1(X)$ and $M_1(Y)$.

Proposition 2.3.5 (Boundary geometry of interior b-maps)**.** *Let $f : X \to Y$ be an interior b-map.*

(a) $f^{-1}(H)$ is a union of boundary hypersurfaces for each $H \in M_1(Y)$. More precisely

$$f^{-1}(H) = \bigcup_{G \in E(H)} G\,, \quad E(H) = \{G : (G, H) \in E\} \tag{2.6}$$

(b) For every face F of X there is a unique face F' of Y so that

$$p \in F^{\mathrm{o}} \implies f(p) \in F'^{\mathrm{o}}$$

We leave the proof as an exercise. Therefore, an interior b-map $f : X \to Y$ induces a map

$$\bar{f} : M(X) \to M(Y)$$

mapping F to F' as in (b). Recall that X is also considered as face of X (if X is connected).

Example 2.3.6. *The map $f : \mathbb{R}^2_+ \to \mathbb{R}_+$, $f(x, y) = x$ is a b-map, and $\bar{f}(x - axis) = \mathbb{R}_+$, $\bar{f}(y - axis) = \bar{f}(\{(0, 0)\} = \{0\}$.*

The map g in Example 2.3.2 cannot be a b-map since $g^{-1}(0) = \{(0, 0)\}$ is not a union of boundary hypersurfaces.

Note that (2.6) implies that, for any $G \in M_1(X)$, $H \in M_1(Y)$,

$$e(G, H) > 0 \iff f(G) \subset H$$

and that in the opposite case $f(G^{\mathrm{o}})$ is disjoint from H. What is the meaning of the number $e(G, H)$? It describes the rate at which $f(p)$ approaches H when $p \in X$ approaches an interior point $p_0 \in G$: roughly, if p has distance $\delta > 0$ from p_0 then $f(p)$ has distance $\approx \delta^{e(G,H)}$ from H.

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