Prof. Dr. Daniel Grieser 16.11.2020

Second problem set 'Singular Analysis' (Blow-ups and resolutions)

We will talk about these on Wednesday, November 25. Please think about them and present your thoughts. If you want to write up and turn in solutions please do, I will read them!

1. Find iterated blow-ups of $\mathbb{R}^2_+$ (with coordinates x, y) that resolve the following function resp. set:
   (a) $f(x,y) = \dfrac{x^2}{x^2 + xy + y^3}$. Interpret the front faces in terms of scales. What are the dominant terms of $x^2 + xy + y^3$ near the corners resp. the interiors of the front faces? Insight?
   (b) $\{x^a = y^b\}$ where $a, b \in \mathbb{N}$ are relatively prime.
2. Prove that $\rho = x + y$, $\sigma = \frac{x-y}{x+y}$ define global coordinates on $[\mathbb{R}^2_+, 0]$, with ρ defining the front face.[1]
3. **(Commuting blow-ups)**
   Let $X = \mathbb{R}^3_+$, $Y =$ the y-axis and $Z = \{0\}$.
   (a) Prove that $[X; Y, Z] = [X; Z, Y]$, e.g. by using projective coordinates.
   (b) Find a *good* proof of $[X; Y, Z] = [X; Z, Y]$ (if you haven't done so in (a)).
4. **(Another commutation theorem)**
   Let $X = \mathbb{R}_{+,x} \times \mathbb{R}^2_{y,z}$, and Y, Z the y- resp. z-axis. We know that $[X; Y, Z] \neq [X; Z, Y]$. Prove that this can be 'healed' by blowing up the intersection either before or afterwards:
   $$[X; Y \cap Z, Y, Z] = [X; Y \cap Z, Z, Y]$$
   $$[X; Y, Z, Y \cap Z] = [X; Z, Y, Y \cap Z]$$
   Also show that these two spaces are not the same.[2] Which one corresponds to the roof of the Galleria Vittorio Emanuele II in Milano (see back)? Can you find an architectural structure representing the other?
5. **(Lifting vector fields)**
   Prove that any linear vector field on $\mathbb{R}^n$ lifts to a b-vector field on $[\mathbb{R}^n, 0]$ by showing that its flow lifts to a smooth flow on $[\mathbb{R}^n, 0]$. Calculate the lift.
   Remark: This gives a third proof of the fact that the vector fields $z_i \partial_{z_j}$ lift to b-vector fields (beyond using scaling and homogenity, as done in the lecture, or calculating in projective coordinates), which implies that any vector field on $\mathbb{R}^n$ vanishing at 0 lifts to a b-vector field on $[\mathbb{R}^n, 0]$.
   See back for hint.

[1] These are a compromise between polar coordinates, which are global but transcendental, and projective coordinates, which are rational but only local.
[2] This holds generally for X a manifold with corners and Y, Z cleanly intersecting p-submanifolds.

Figure 1: Galleria Vittorio Emanuele II, Milano. License: CC BY 2.0, https://commons.wikimedia.org/w/index.php?curid=40952131, Paul Bica, Toronto, Canada,

Hint for 5.: If the vector field is $V_z = Az$ for an $n \times n$-matrix A then its flow is $\Phi(t, z) = e^{tA}z$.