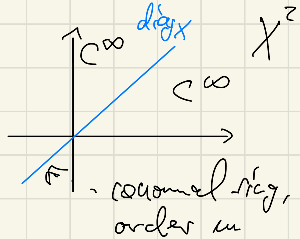
2021-01-13

X manifold, $\Psi^m(X)$
$m \in \mathbb{R}$

X compact



(Algebra 2) ("Exact") $\Rightarrow \forall P \in \Psi^m$ elliptic $\exists Q \in \Psi^{-m}$

$$PQ = I + R, \quad QP = I + R' \qquad R, R' \in \Psi^{-\infty}$$

(Map) $\qquad P: H^s \to H^{s-m} \qquad \forall s$

What does the ΨDO calculus do for you?

1) Elliptic regularity:

P elliptic, $\qquad Pu = f \in H^s \Rightarrow u \in H^{s+m}$
$u \in H^t$ (some t)

Proof: $\qquad \underbrace{Qf}_{H^{s-(-m)}} = QPu = u + \underbrace{R'u}_{\in C^\infty \subset H^{s+m}}$
(Map for Q)

$\Rightarrow u \in H^{s+m}$ q.e.d.

2) Fredholm property

P elliptic $\Rightarrow P: H^s \to H^{s-m}$ Fredholm $\forall s$

i.e.: · $\dim \ker P < \infty$
· $\dim \operatorname{coker} P$, $\operatorname{coker} P = H^{s-m} / \operatorname{Ran} P$

Pf: $R, R' \in \Psi^{-\infty} \Rightarrow$ they are compact as op's $H^s \to H^s \quad \forall s$

Func. ana: $\exists$ parametrix Q with compact remainders $\Rightarrow P$ Fredholm.

Sometimes we know that P is invertible by some func. ana. argument, e.g. for $P = -\Delta + 1$:
$-\Delta \geq 0 \Rightarrow -\Delta + 1$ invertible.

[invertible as operator $H^s \to H^{s-m}$, for some (and hence any) s.
eg $\quad H^m \to H^0 = L^2$]

3) Precise description of (Fredholm) inverse

• P elliptic and invertible then $P^{-1} \in \Psi^{-m}$
$$P^{-1} - Q \in \Psi^{-\infty}$$

Proof: $PQ = I + R \Rightarrow Q = \underbrace{P^{-1}}_{} + P^{-1}R$

$QP = I + R' \Rightarrow Q = P^{-1} + R'\underbrace{P^{-1}}_{}$

$\Rightarrow Q = P^{-1} + R'(Q - P^{-1}R)$

$\Rightarrow \underbrace{Q - P^{-1}}_{\Psi^{-m}} = \underbrace{R'Q}_{\Psi^{-\infty}} - \underbrace{R'P^{-1}R}_{\in \Psi^{-\infty}}$

Proof: Use $R \in \Psi^{-\infty} \Leftrightarrow R: \mathcal{D}' \to C^\infty$ continuous

Then
$$\begin{tikzcd} \mathcal{D}' \arrow[r, "R"] \arrow[rrr, bend right, dotted, "R'P^{-1}R"'] & C^\infty \subset L^2 \arrow[r, "P^{-1}"] & L^2 \subset \mathcal{D}' \arrow[r, "R'"] & C^\infty \end{tikzcd}$$

qed.

„Biideal property"

More generally:

Def: S Fredholm inverse of P $:\Leftrightarrow$
$$PS = I - \Pi, \quad SP = I - \Pi'$$
where

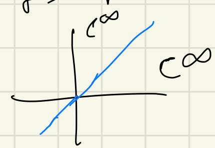
Π is a continuous projection to a C^∞ complement of $\operatorname{Ran} P$

Π' is a continuous projection to $\ker P$.

Then: S Fredholm inverse $\Rightarrow S \in \Psi^{-m}$, $S - Q \in \Psi^{-\infty}$

Ex:
$$P: \begin{tikzcd} H^s \arrow[rr] \arrow[d, equal] & & H^{s-m} \arrow[d, equal] \\ \ker P \arrow[d, phantom, "\oplus"] & 0 & (\operatorname{Ran} P)^\perp \arrow[l, "S"'] \arrow[d, phantom, "\oplus"] \\ (\ker P)^\perp \arrow[rr, dotted, "P"] & & \operatorname{Ran} P \arrow[ll, bend left, dotted, "S"] \end{tikzcd}$$

Important: this says that we know the singularity of (the Schwartz kernel of) P^{-1} to any order.



constructive (algebraic) way to get it

We don't know the smooth part of P^{-1}

General principle:

We have a chance of describing algebraically solutions at singularities or in limiting regimes, but usually not elsewhere.

$\boxed{\text{ΨDO calculus is "constructive"}}$

Also: Extension to vector bundles is immediate.

$$\begin{array}{ccc} E & & F \\ \downarrow & \swarrow & \\ X & & \end{array} \qquad P : C^\infty(X,E) \to C^\infty(X,F)$$

ΨDO and model operators $(X=\mathbb{R}^n)$

Recall in coord. $P=\sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$, $\sigma_{tot,P}(x,\xi)=\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$

⇝ Schwartz kernel of P:
$$P(x,x') = \int e^{i(x-x')\xi}\,\sigma_{tot,P}(x,\xi)\,d\xi$$

⇝ (find) parametrix: $Q(x,x') = \int e^{i(x-x')\xi}\,\frac{1}{\sigma_P(x,\xi)}\,d\xi$

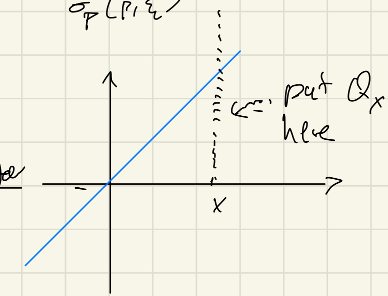
$\sigma_P(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ princ. symbol.

Note: If $P_p = \sum_{|\alpha|=m} a_\alpha(p) D_x^\alpha$ then $Q_p = P_p^{-1}$

has S.k. $Q_p(x,x') = \int e^{i(x-x')\xi}\,\frac{1}{\sigma_P(p,\xi)}\,d\xi$

So $Q(x,x') = Q_x(x,x')$

We call P_p the model operator of P at p



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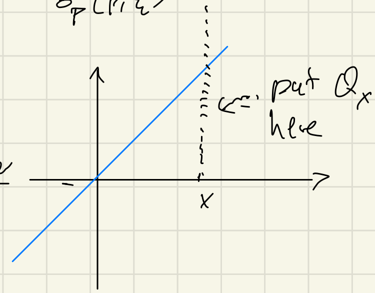


We call P_p the model operator of P at p

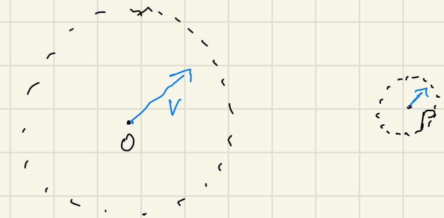
Summary: Get parametrix Q of P as follows:

$$\begin{array}{ccc} P & \longrightarrow & \text{model operators } P_p,\ p\in X \\ & & \Big\downarrow \text{invert (symbol, Fourier trf.)} \\ Q & \xleftarrow[\text{Schwartz kernels}]{\text{glue as}} & \text{inverses } Q_p \end{array}$$

(This gives $PQ = I + R$, $R \in \Psi^{-1}$, $QP = I + R'$, $R' \in \Psi^{-1}$

use algebra property / iteration to improve to $R \in \Psi^{-\infty}$)

How to arrive at model operators P_p: Zoom in at p



Microscope map for $\varepsilon > 0$: $M_\varepsilon : v \mapsto p + \varepsilon v = x$

Zoom in = pull-back under M_ε:

(Note $M_\varepsilon^* \partial_{x_j} = \varepsilon^{-1} \partial_{v_j}$)

$$\Rightarrow M_\varepsilon^* P = \sum_{|\alpha| \le m} a_\alpha(p + \varepsilon v)\, \varepsilon^{-|\alpha|} D_v^{|\alpha|}$$

$$\Rightarrow \boxed{P_p = \lim_{\varepsilon \to 0} \varepsilon^m M_\varepsilon^* P}$$

Note: On manifold X, P_p is naturally defined on T_pX.

4. The b-calculus

- read lec. notes 1.2 again
- ref's: Melrose: Atiyah-Patodi-Singer ... book
  DG: Basics of b-calculus.

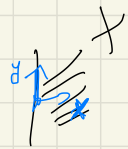
Two meanings:

- general term all these techniques for singular problems
- here: specific type of singularity / degeneration

Setting: · X manifold with compact boundary

· $\mathcal{V}_b(X) = \{\text{vector fields tangent to } \partial X\}$
$= \operatorname{span}\{x\partial_x, \partial_y\}$ locally

$y = (y_1, \ldots, y_{n-1})$



· $\mathrm{Diff}_b^m(X) = \{ a + \sum b_{i_1 \ldots i_r} V_{i_1} \cdots V_{i_r} \quad V_{i_j} \in \mathcal{V}_b, \ r \le m \}$

$= \{ \sum a_{k\alpha}(x,y) (xD_x)^k D_y^\alpha \}$ locally near ∂X

Goal: Construct Ψ_b^m:

- $\mathrm{Diff}_b^m \subset \Psi_b^m$ if $m \in \mathbb{N}_0$
- Ψ_b^* contains parametrices/inverses of elliptic/(ell. + invertible) elements of Diff_b^m.

and allows to prove elliptic regularity: $Pu = f$

- interior regularity
- boundary regularity: conormal or polyhomogeneous

· what does "elliptic" mean?

Outlook: Two steps:

- small calculus: Ψ_b^m similar to Ψ^m, invert model operator at $p \in \mathring{X}$, uniform as $p \to \partial X$

This gives a "small parametrix" for elliptic P which yields interior + conormal regularity, but not: phg reg., Fredholm property.

- full calculus $\Psi_b^{m,\mathcal{E}}$: invert a model operator at ∂X

gives phg reg., Fredholm.

Geometric settings where b-problems arise:

- conical metrics on man. w. bd. X:

  positive semi-definite symmetric 2-tensor on X:

  $p: \quad g_p: T_pX \times T_pX \to \mathbb{R}$.

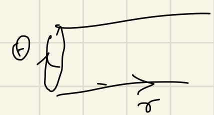
  - positive definite in $\mathring{X}$
  - near ∂X: $\quad g = dx^2 + x^2 h$

    h smooth symm. 2-tensor, pos. def. on $T\partial X$.

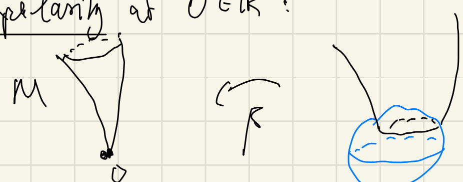
Ex: · $\mathbb{R}^2$, polar coord's r, θ

$\Rightarrow g_{eucl} = dr^2 + r^2 d\theta^2$

on $\underset{r}{\mathbb{R}_+} \times \underset{\theta}{S^1}$



- Similar on $\mathbb{R}^n$
- Similar on any space $M \subset \mathbb{R}^n$ with a conic singularity at $0 \in \mathbb{R}^n$:



$X := \beta^+ M$

Take any Riem. Metric on $\mathbb{R}^n$, pull-back under β and restrict to X.

Fact: the Laplace-Beltrami operator Δ for a conical metric is

$$\Delta = x^{-2} P, \quad P \in \mathrm{Diff}_b^2(X).$$

Ex: $\mathbb{R}^2$: $\quad \Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$

$$= r^{-2} \underbrace{\left(r^2\partial_r^2 + r\partial_r + \partial_\theta^2\right)}_{P}$$

$$P = (r\partial_r)^2 + \partial_\theta^2.$$