2020-11-11

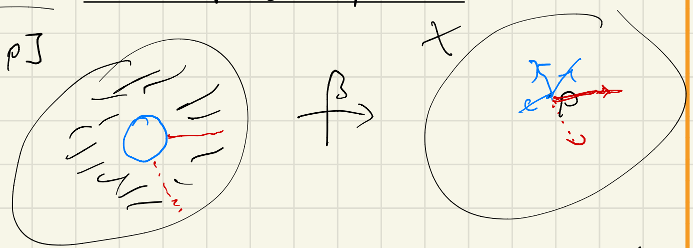
# II.5 Blow-up

$X$ mwc (manifold with corners)
$Y$ p-submanifold

$\rightsquigarrow \underbrace{[X, Y]}_{\text{blow-up of } X \text{ in } Y} \xrightarrow{\beta} X$ $\qquad \beta$: blow-down map

## II.5.1 Blow-up of a point

$[X, p]$




$X = \mathbb{R}^n$, $p = 0$: $\quad [\mathbb{R}^n, 0]_{mod} = \mathbb{R}_+ \times S^{n-1}$

$$\begin{array}{ccc} [\mathbb{R}^n, 0]_{mod} & & (r, \omega) \\ \downarrow & & \downarrow \\ \mathbb{R}^n & & r \cdot \omega \end{array}$$

Let $X$ be a manifold, $p \in X$.
Tangent space at $p$: $T_pX$.
Def: $S_pX := (T_pX \setminus 0) / \mathbb{R}_{>0}$ where $\mathbb{R}_{>0}$ acts on $T_pX$ by $t \cdot v := tv$

$S_pX$ = { "directions at $p$" }
"sphere at $p$" but without metric.

Def: $[X, p] := (X \setminus p) \sqcup S_pX$

$$\begin{array}{ccccc} [X,p] & & X\setminus p & & S_pX \\ \downarrow \beta & & \downarrow id & & \downarrow \\ X & & X \setminus p & & p \end{array}$$

We turn this into a manifold with boundary as follows:
Choose local chart $\quad \varphi: \mathbb{R}^n \to U \subset X$
$\qquad 0 \mapsto p$

Then $d\varphi_0: T_0\mathbb{R}^n \to T_pX$
induces map $d\varphi_0: S_0\mathbb{R}^n \to S_pX$.

Then $(r, \omega) \longmapsto \begin{cases} r\omega \in \mathbb{R}^n \setminus 0 & \text{if } r > 0 \\ [\omega] \in S_0\mathbb{R}^n & \text{if } r = 0 \end{cases}$

$$\begin{array}{ccc} \mathbb{R}_+ \times S^{n-1} & \longrightarrow & (\mathbb{R}^n \setminus 0) \sqcup S_0\mathbb{R}^n \\ & & \varphi \downarrow \quad \downarrow \quad \downarrow d\varphi_0 \\ & & (U \setminus p) \sqcup S_pX \end{array}$$

the lemma on lifting diffeos implies:

$\forall\, F: X \to X'$ diffeo, $p \mapsto p'$

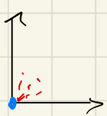
then $\exists!\, \tilde F$:

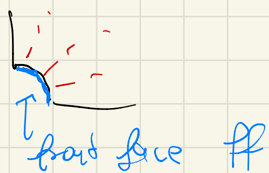
$$\begin{tikzcd} {[X,p]} \arrow[r, "\tilde F"] \arrow[d, "\beta"] & {[X',p']} \arrow[d, "\beta'"] \\ X \arrow[r, "F"] & X' \end{tikzcd}$$

If $X$ is a manifold with corners, $p \in X$, then

$[X,p] := (X \setminus p) \sqcup (S_p^+ X)$

$S_p^+ X :=$ inward pointing directions.



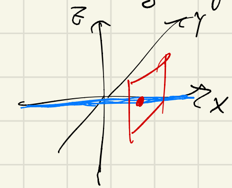


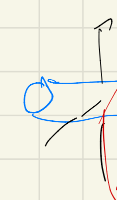

## II.5.2 Blow-up of a p-submanifold

This is like introducing cylindrical coordinates.

$X = \mathbb{R}^3$

$Y = x$-axis




① Local model: $Y = \mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n$
$(z', z'') = z$

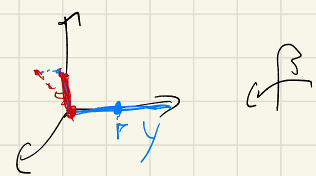
so $Y = \{z'' = 0\}$.

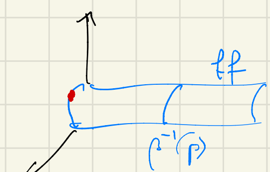
Define $[\mathbb{R}^n, Y] = [\mathbb{R}^m \times \mathbb{R}^{n-m}, \mathbb{R}^m \times \{0\}]$
$:= \mathbb{R}^m \times [\mathbb{R}^{n-m}, \{0\}]$

② Local model with corners:
$X = \mathbb{R}^n_k = \underset{x}{\mathbb{R}^k_+} \times \underset{y}{\mathbb{R}^{n-k}}$, $Y = \{x_i = 0,\ i \in I;\ y_j = 0,\ j \in J\}$

Then $X = Y \times W$, then define $[X, Y]$ as before.

$\mathbb{R}^3_3$




$[X, Y]$

Invariance / lifting lemma: diffeo fixing $Y$ pointwise lifts to $[X, Y]$.

(3) $X$ mwc, $Y \subset X$ p-submanifold.

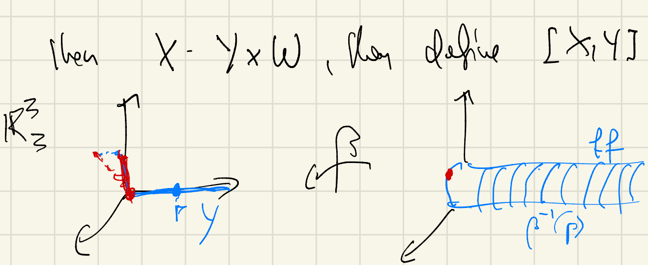
Define $[X, Y]$ by gluing local models.

Invariantly: $N_p^+ Y = T_p^+ X / T_p Y$, $p \in Y$

$$[X, Y] := (X \setminus Y) \sqcup \left(N^+ Y / \mathbb{R}_{>0}\right)$$

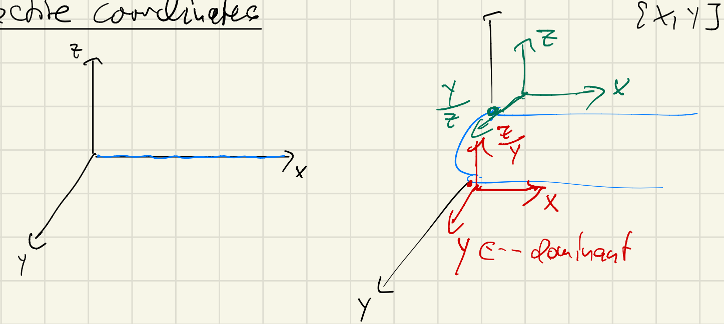
(dilation action)

Then $X = Y \times W$, then define $[X, Y]$ as before




Rem: $ff := \beta^{-1}(Y)$. The map $\beta : ff \to Y$ is a fibration of ff.

Projective coordinates

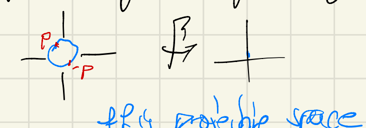



don't change the axial variable.

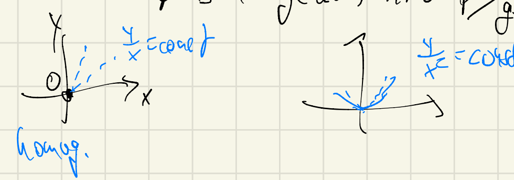
Rem: • non-oriented blow-up (in algebraic geometry)

$[\mathbb{R}^2, 0]_{non \cdot or}$ :

$:= [\mathbb{R}^2, 0] / (p \sim -p \; \forall p)$




• quasihomogeneous blow-up / generalized blow-up.

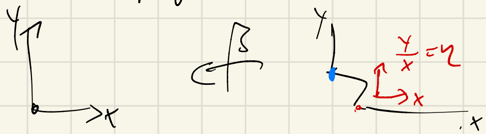



x: weight 1

y: weight 2

Rem: The blow-down map is a b-map.

Pf: Check in projective coordinates.




$$\beta^* x = x \quad , \quad \beta^* y = x \cdot \eta$$

+ similar in • coord's $\xi = \frac{x}{y}, y$.

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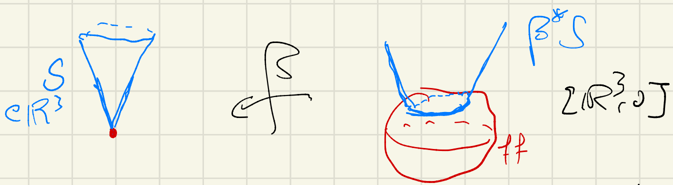
## II.5.3 Why do blow-ups? Lifts and resolutions

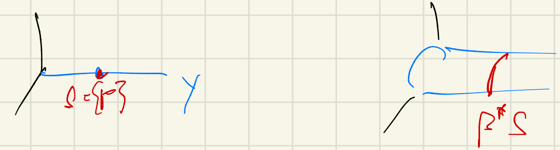
Using blow-ups one can resolve singular objects.

- Subsets:




Def: Let $X$ be a mwc, $Y \subset X$ a p-submfd, $S \subset X$ connected. The lift of $S$ under the blowup of $X$ in $Y$ is

$$\beta^* S := \begin{cases} \beta^{-1}(S \setminus Y) & \text{if } S \not\subset Y \\ \beta^{-1}(S) & \text{if } S \subset Y. \end{cases} \qquad (\beta: [X, Y] \to X)$$




Def: Iterated blow-up: If $Y_1 \subset X$ p-submfd, $Y_2 \subset [X, Y_1]$ p-submfd etc. $Y_3 \subset [[X, Y_1], Y_2]$

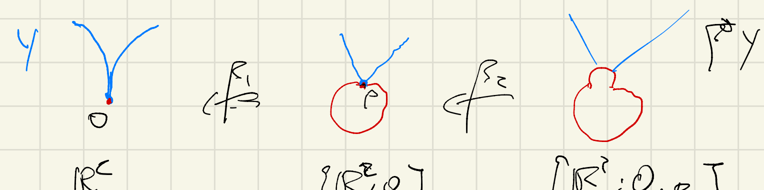
then $[X; Y_1, Y_2, \dots, Y_\ell] := [[[X, Y_1], Y_2] \dots Y_\ell]$.

and $\beta^* S := \beta_s^* \dots \beta_2^*(\beta_1^* S)$

$\beta_1 : [X, Y_1] \to X$ etc.

$S$ is resolved by $\beta$ if $\beta^* S$ is a p-submanifold of $[X; Y_1, \dots, Y_s]$.

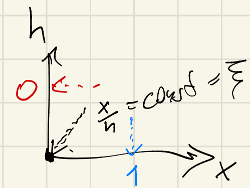
example: $X = \mathbb{R}^2$, $Y$ = finite cusp (shown)



**Thm** (Hironaka?): $S \subset \mathbb{R}^n$ algebraic
$\Rightarrow$ $S$ can be resolved by iterated blow-up.

## Resolving functions

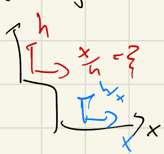
**Def:** $X$ mwc, $f: X \to \mathbb{C}$.
$f$ is smoothly resolved by iterated blow-up if $\beta^* f$ is smooth.




**Ex:** $f(x,h) = \dfrac{x}{x+h}$ $(x,h>0)$

$$= \frac{x/h}{x/h+1} = \frac{\xi}{\xi+1} \to \frac{\xi}{\xi+1} \text{ as } h \to 0$$

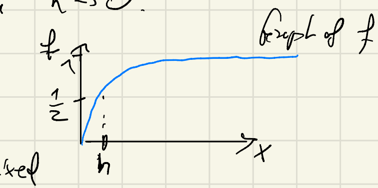
Claim: $f$ is resolved by blow-up of $(0,0)$:




$$\beta^* f = \begin{cases} \dfrac{\xi}{\xi+1} \\ \dfrac{1}{1+y} \end{cases} \Rightarrow \beta^* f \text{ is smooth}$$

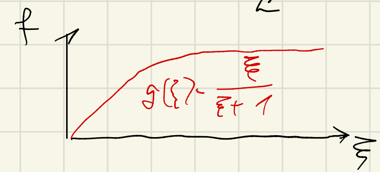
Question: how can I understand the behavior of $f_h(x) := f(x,h)$ as $h \to 0$.

$f_h(x) = \dfrac{x}{x+h}$

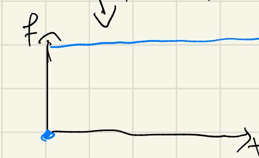



a scale $x \sim h$

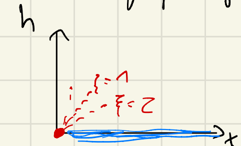
$h \to 0$, $\frac{x}{h} = \xi$ fixed



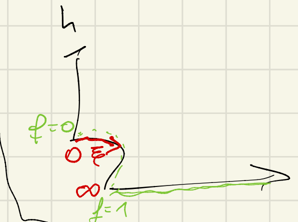

$h \to 0$, $x$ fixed




How to find this the graph of $f(x,h)$?




blow-up makes $g$ visible:




different boundary hypersurfaces of a resolved space correspond to different scales (looking at limit $h \to 0$): here: ff $\hat{=}$ scale $x \sim h$
x-axis $\hat{=}$ scale $x \sim 1$

Why is this useful?

Matched asymptotic expansions can be made very clear, rigorous using blow-ups.