Singular analysis

I. Introduction: What is singular analysis?

I.1 What's a singularity?

- "A place where a mathematical object behaves differently than at most places."
- "Singular is the opposite of regular."

Regularity usually involves existence of a simple local model.

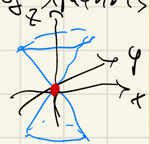
Spaces: regular ≈ smooth manifold (C^∞)

local model: $\mathbb{R}^n$

How do singular spaces arise?

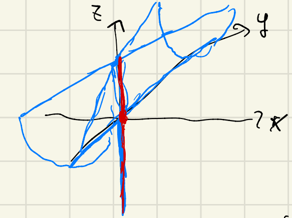
a) As level sets of maps = solution sets of equations

- In $\mathbb{R}^3$: $\{(x,y,z): x^2+y^2=z^2\}$



- In $\mathbb{R}^3$: Whitney umbrella: $x^2=y^2z$

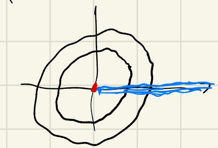
$x^2=y^2z$:



b) As quotients by non-free group actions.

Ex: S^1 acts on $\mathbb{R}^2$ by rotation.
$\{z\in\mathbb{C}: |z|=1\} \cong \mathbb{R}/\mathbb{Z}$

$\mathbb{R}^2/S^1$ = set of orbits $= [0,\infty)$



c) Solutions of geometric PDE.

Ex: Einstein's equation. Schwarzschild space-time has a singularity.

Remark: Often it makes sense to consider compact manifolds as regular.

Ex: $\mathbb{R}^n$ is "singular at ∞"

Smooth maps

<u>Recall</u>: $f : \Omega \subset \mathbb{R}^n \longrightarrow \Omega' \subset \mathbb{R}^k$ smooth.

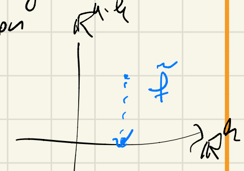
$p \in \Omega$ regular point $:\Longleftrightarrow df|_p : \mathbb{R}^n \to \mathbb{R}^k$ surjective

$q \in \Omega'$ regular value $:\Longleftrightarrow$ All $p \in f^{-1}(q)$ are regular

<u>Fact</u>: · q regular value $\Longrightarrow f^{-1}(q)$ is submanifold

· p regular point $\Longleftrightarrow f$ is locally like a projection

(implicit function theorem)



(smooth) Vector fields:

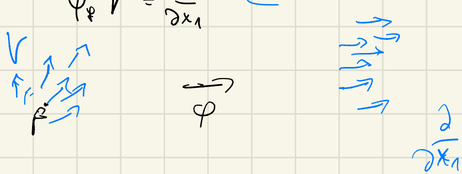
$V : \Omega \overset{\text{open}}{\subset} \mathbb{R}^n \longrightarrow \mathbb{R}^n$ smooth.

$p \in \Omega$ singular point of $V :\Longleftrightarrow V(p) = 0$.

Why? If $V(p) \neq 0$ then $\exists$ nbhd. U of p and a diffeomorphism $\varphi : U \to U' \subset \mathbb{R}^n$ so that

$$\varphi_* V = \frac{\partial}{\partial x_1}$$

local model



(compare Hilbert's 16th problem)

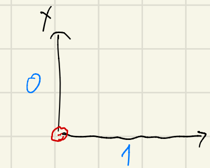
Functions:

· in $\mathbb{C}$ (isolated sing.): removable, pole, essential

· Distributions

· $f(x,y) = \dfrac{x}{x+y}$ on $\mathbb{R}^2_+ \setminus 0$:

$\mathbb{R}_+ = [0, \infty)$



· <u>Differential operators</u> (linear)

ODE: $P = \sum_{k=0}^{m} a_k(x)\,\partial_x^k$, $x \in \mathbb{R}$, $\partial_x = \frac{\partial}{\partial x} = \frac{d}{dx}$

$x_0 \in \mathbb{R}$ regular point for $P :\Longleftrightarrow a_m(x_0) \neq 0$.

If $a_m(x_0) \neq 0$ then $\exists!$ smooth solution of $Pu = 0$ with given initial data at x_0.

If not: <u>ex</u>: $P = x\partial_x - c$. $\quad x_0 = 0$

$(x\partial_x - c)u = 0 : \quad xu' = c \cdot u.$

$u(x) = A \cdot x^c$ not smooth at $x = 0$ if $c \notin \mathbb{N}_0$.

PDEs:

ex: Δ on $\mathbb{R}^2$ in polar coordinates (r,θ):

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 = \frac{1}{r^2}\underbrace{\left[r^2\partial_r^2 + r\partial_r + \partial_\theta^2\right]}_{\tilde P}$$

(not elliptic at $r=0$)

ex: behavior at ∞:

$$s = \frac{1}{r} \Rightarrow \frac{\partial}{\partial r} = \frac{\partial s}{\partial r}\frac{\partial}{\partial s} = -\frac{1}{r^2}\frac{\partial}{\partial s} = -s^2\frac{\partial}{\partial s}$$

$$\Rightarrow \Delta = (s^2\partial_s)^2 - s^3\partial_s + s^2\partial_\theta^2$$ elliptic for $s>0$, not for $s=0$.

$$(s^2\partial_s)^2 u = s^2\partial_s(s^2\partial_s u) = s^4\partial_s^2 u + 2s^3\partial_s u$$

$$\Rightarrow \Delta = s^2\underbrace{\left[s^2\partial_s^2 + s\partial_s + \partial_\theta^2\right]}_{=\tilde P}$$

$\rightsquigarrow$ $s=0$ ($\hat{=}\ r=\infty$) behaves similarly to $r=0$.

(with respect to: $\Delta u = 0$, behavior of u as $r\to 0$ or $r\to\infty$)

- In applications: crack theory

vibrations:

- Coulomb Schrödinger eq'n: $\Delta + \frac{1}{|x|}$ in $\mathbb{R}^3$

II.2 Goals and methods

Main goal: Understand solutions of „singular PDE".

Some central ideas:

- Fourier transform for const coefficients
- idea of model problems
  - freeze coeff. at regular pts.
  - other models at ~~...~~ sing. pts.
- put problem into a system $\rightsquigarrow$ involves geometric resolution (blow-ups)
- our regular space are manifolds with corners.